NEW TECHNIQUES FOR POINTED HOPF ALGEBRAS

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Abstract. We present techniques that allow to decide that the dimension of some pointed Hopf algebras associated with non-abelian groups is infinite. These results are consequences of [AHS]. We illustrate each technique with applications.

Dedicado a Isabel Dotti y Roberto Miatello en su sexagésimo cumpleaños.

Introduction

0.1. Let G be a finite group and let ${}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ be the category of Yetter-Drinfeld modules over $\mathbb{C}G$. The most delicate of the questions raised by the Lifting Method for the classification of finite-dimensional pointed Hopf algebras H with $G(H) \simeq G$ [AS1, AS3], is the following:

> *Given $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$, decide when the Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional.*

Recall that a Yetter-Drinfeld module over the group algebra $\mathbb{C}G$ (or over G for short) is a left $\mathbb{C}G$-module and left $\mathbb{C}G$-comodule M satisfying the compatibility condition $\delta(g.m) = ghg^{-1} \otimes g.m$, for all $m \in M_h$, $g, h \in G$. The list of all objects in ${}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ is known: any such is completely reducible, and the class of irreducible ones is parameterized by pairs $(\mathcal{O}, \rho)$, where $\mathcal{O}$ is a conjugacy class in G and ρ is an irreducible representation of the isotropy group G^s of a fixed $s \in \mathcal{O}$. We denote the corresponding Yetter-Drinfeld module by $M(\mathcal{O}, \rho)$.

In fact, our present knowledge of Nichols algebras is still preliminary. However, an important remark is that the Nichols algebra $\mathfrak{B}(V)$ depends (as algebra and coalgebra) just on the underlying braided vector space (V, c)– see for example [AS3]. This observation allows to go back and forth between braided vector spaces and Yetter-Drinfeld modules. Indeed, the same braided vector space could be realized as a Yetter-Drinfeld module over different groups, and even in different ways over the same group, or not at all. The braided vector spaces that do appear as Yetter-Drinfeld modules over some finite group are those coming from racks and 2-cocycles [AG].

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Thus, a comprehensive approach to the question above would be to solve the following:

> *Given a braided vector space (V, c) determined by a rack and a 2-cocycle, decide when* $\dim \mathfrak{B}(V) < \infty$.

But at the present moment and with the exception of the diagonal case mentioned below, we know explicitly very few Nichols algebras of braided vector spaces determined by racks and 2-cocycles; see [FK, MS, G1, AG, G2].

0.2. The braided vector spaces that appear as Yetter-Drinfeld modules over some finite *abelian* group are the diagonal braided vector spaces. This leads to the following question: *Given a braided vector space (V, c) of diagonal type, decide when the Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional.* The full answer to this problem was given in [H2], see [AS2, H1] for braided vector spaces of Cartan type– and [AS4] for applications. These results on Nichols algebras of braided vector spaces of diagonal type were in turn used for more general pointed Hopf algebras. Let us fix a non-abelian finite group G and let $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ irreducible. If the underlying braided vector space contains a braided vector subspace of diagonal type, whose Nichols algebra has infinite dimension, then $\dim \mathfrak{B}(V) = \infty$. In turns out that, for several finite groups considered so far, many Nichols algebras of irreducible Yetter-Drinfeld modules have infinite dimension; and there are short lists of those not attainable by this method. See [G1, AZ, AF1, AF2, FGV].

0.3. An approach of a different nature, inspired by [H1], was presented in [AHS]. Let us consider $V = V_1 \oplus \cdots \oplus V_\theta \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$, where the V_i's are irreducible. Then the Nichols algebra of V is studied, under the assumption that the $\mathfrak{B}(V_i)$ are known and finite-dimensional, $1 \le i \le \theta$. Under some circumstances, there is a Coxeter group $\mathcal{W}$ attached to V, so that $\mathfrak{B}(V)$ finite-dimensional implies $\mathcal{W}$ finite. Although the picture is not yet complete, the previous result implies that, for a few G– explicitly, $\mathbb{S}_3$, $\mathbb{S}_4$, $\mathbb{D}_n$– the Nichols algebras of some V have infinite dimension. These applications rely on the lists mentioned at the end of 0.2.

0.4. The purpose of the present paper is to apply the results in 0.3 to discard more irreducible Yetter-Drinfeld modules. Namely, let $V = V_1 \oplus V_2 \in {}^{\mathbb{C}\Gamma}_{\mathbb{C}\Gamma}\mathcal{YD}$, where $\Gamma = \mathbb{S}_3$, $\mathbb{S}_4$ or $\mathbb{D}_n$, such that $\dim \mathfrak{B}(V) = \infty$ by [AHS, Section 4]. Then there is a rack $(X, \triangleright)$ and a cocycle $\mathfrak{q}$ such that $(V, c) \simeq (\mathbb{C}X, c_{\mathfrak{q}})$. Let G be a finite group, let $\mathcal{O}$ be a conjugacy class in G, $s \in \mathcal{O}$, $\rho \in \widehat{G^s}$ and $M(\mathcal{O}, \rho) \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ the irreducible Yetter-Drinfeld module corresponding to $(\mathcal{O}, \rho)$. We give conditions on $(\mathcal{O}, \rho)$ such that $M(\mathcal{O}, \rho)$ contains a braided vector subspace isomorphic to $(\mathbb{C}X, c_{\mathfrak{q}})$; thus, necessarily, $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$. We illustrate these new techniques with several examples; see in particular Example 3.9 for one that can not be treated via abelian subracks.

0.5. The facts glossed in the previous points strengthen our determination to consider families of finite groups, in order to discard those irreducible Yetter-Drinfeld modules over them with infinite-dimensional Nichols algebra by the 'subrack method'. Natural candidates are the families of simple groups, or closely related; cf. the classification of simple racks in [AG]. The case of symmetric and alternating groups is treated in [AZ, AF1, AF2, AFZ]; Mathieu groups in [F1]; other sporadic groups in [AFGV]; some finite groups of Lie type with rank one in [FGV, FV]. Particularly, a list of only 9 types of pairs $(\mathcal{O}, \rho)$ for $\mathbb{S}_m$ whose Nichols algebras might be finite-dimensional is given in [AFZ]; an analogous list of 7 pairs out of 1137 (for all 5 Mathieu simple groups) is given in [F1]; the sporadic groups J_1, J_2, J_3, He and Suz are shown to admit no non-trivial pointed finite-dimensional Hopf algebra in [AFGV]. Our new techniques are crucial for these results.

0.6. If for some finite group G there is at most one irreducible Yetter-Drinfeld module V with finite-dimensional Nichols algebra, then [AHS, Th. 4.2] can be applied again. If the conclusion is that $\dim \mathfrak{B}(V \oplus V) = \infty$, then we can build a new rack together with a 2-cocycle realizing $V \oplus V$, and investigate when a conjugacy class in another group G' contains this rack, and so on.

1. Notations and conventions

The base field is $\mathbb{C}$ (the complex numbers).

1.1. **Braided vector spaces.** A *braided vector space* is a pair (V, c), where V is a vector space and $c : V \otimes V \to V \otimes V$ is a linear isomorphism such that c satisfies the braid equation: $(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$.

Let V be a vector space with a basis $(v_i)_{1 \le i \le \theta}$, let $(q_{ij})_{1 \le i,j \le \theta}$ be a matrix of non-zero scalars and let $c : V \otimes V \to V \otimes V$ be given by $c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i$. Then (V, c) is a braided vector space, called of *diagonal type.*

Examples of braided vector spaces come from racks. A *rack* is a pair $(X, \triangleright)$ where X is a non-empty set and $\triangleright : X \times X \to X$ is a function– called the multiplication, such that $\phi_i : X \to X$, $\phi_i(j) := i \triangleright j$, is a bijection for all $i \in X$, and

$$i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k) \qquad \text{for all } i,\ j,\ k \in X. \tag{1.1}$$

For instance, a group G is a rack with $x \triangleright y = xyx^{-1}$. In this case, $j \triangleright i = i$ whenever $i \triangleright j = j$ and $i \triangleright i = i$ for all $i \in G$. We are mainly interested in subracks of G, e. g. in conjugacy classes in G.

Let $(X, \triangleright)$ be a rack. A function $\mathfrak{q} : X \times X \to \mathbb{C}^{\times}$ is a *2-cocycle* if $q_{i,j \triangleright k}\, q_{j,k} = q_{i \triangleright j, i \triangleright k}\, q_{i,k}$, for all i, j, $k \in X$. Then $(\mathbb{C}X, c_q)$ is a braided vector space, where $\mathbb{C}X$ is the vector space with basis e_k, $k \in X$, and the braiding is given by

$$c_q(e_k \otimes e_l) = q_{k,l}\, e_{k \triangleright l} \otimes e_k, \text{ for all } k, l \in X.$$

A subrack T of X is *abelian* if $k \triangleright l = l$ for all $k, l \in T$. If T is an abelian subrack of X, then $\mathbb{C}T$ is a braided vector subspace of $(\mathbb{C}X, c_q)$ of diagonal type.

Definition 1.1. Let X be a rack. Let X_1 and X_2 be two disjoint copies of X, together with bijections $\varphi_i : X \to X_i$, $i = 1, 2$. The *square* of X is the rack with underlying set the disjoint union $X_1 \coprod X_2$ and with rack multiplication

$$\varphi_i(x) \triangleright \varphi_j(y) = \varphi_j(x \triangleright y),$$

$x, y \in X$, $1 \leq i, j \leq 2$. We denote the square of X by $X^{(2)}$. This is a particular case of an amalgamated sum of racks, see e. g. [AG].

1.2. **Yetter-Drinfeld modules.** We shall use the notation given in [AF1]. Let G be a finite group. We denote by $|g|$ the order of an element $g \in G$; and by $\widehat{G}$ the set of isomorphism classes of irreducible representations of G. We shall often denote a representant of a class in $\widehat{G}$ with the same symbol as the class itself.

Here is an explicit description of the irreducible Yetter-Drinfeld module $M(\mathcal{O}, \rho)$. Let $t_1 = s, \dots, t_M$ be a numeration of $\mathcal{O}$ and let $g_i \in G$ such that $g_i \triangleright s = t_i$ for all $1 \leq i \leq M$. Then $M(\mathcal{O}, \rho) = \oplus_{1 \leq i \leq M} g_i \otimes V$, where V is the vector space affording the representation ρ. Let $g_i v := g_i \otimes v \in M(\mathcal{O}, \rho)$, $1 \leq i \leq M$, $v \in V$. If $v \in V$ and $1 \leq i \leq M$, then the action of $g \in G$ is given by $g \cdot (g_i v) = g_j(\gamma \cdot v)$, where $g g_i = g_j \gamma$, for some $1 \leq j \leq M$ and $\gamma \in G^s$, and the coaction is given by $\delta(g_i v) = t_i \otimes g_i v$. Then $M(\mathcal{O}, \rho)$ is a braided vector space with braiding $c(g_i v \otimes g_j w) = g_h(\gamma \cdot w) \otimes g_i v$, for any $1 \leq i, j \leq M$, $v, w \in V$, where $t_i g_j = g_h \gamma$ for unique h, $1 \leq h \leq M$ and $\gamma \in G^s$. Since $s \in Z(G^s)$, the center of G^s, the Schur Lemma implies that

$$s \text{ acts by a scalar } q_{ss} \text{ on } V. \tag{1.2}$$

Lemma 1.2. *If U is a subspace of W such that $c(U \otimes U) = U \otimes U$ and $\dim \mathfrak{B}(U) = \infty$, then $\dim \mathfrak{B}(W) = \infty$.* □

Lemma 1.3. [AZ, Lemma 2.2] *Assume that s is* real *(i. e. $s^{-1} \in \mathcal{O}$). If $\dim \mathfrak{B}(\mathcal{O}, \rho) < \infty$, then $q_{ss} = -1$ and s has even order.* □

Let $\sigma \in \mathbb{S}_m$ be a product of n_j disjoint cycles of length j, $1 \leq j \leq m$. Then the type of σ is the symbol $(1^{n_1}, 2^{n_2}, \dots, m^{n_m})$. We may omit j^{n_j} when $n_j = 0$. The conjugacy class $\mathcal{O}_\sigma$ of σ coincides with the set of all permutations in $\mathbb{S}_m$ with the same type as σ; we may use the type as a subscript of a conjugacy class as well. If some emphasis is needed, we add a superscript m to indicate that we are taking conjugacy classes in $\mathbb{S}_m$, like $\mathcal{O}_j^m$ for the conjugacy class of j-cycles in $\mathbb{S}_m$.

2. A technique from the dihedral group $\mathbb{D}_n$, n odd

Let $n > 1$ be an odd integer. Let $\mathbb{D}_n$ be the dihedral group of order $2n$, generated by x and y with defining relations $x^2 = e = y^n$ and $xyx = y^{-1}$. Let $\mathcal{O}_x$ be the conjugacy class of x and let $\operatorname{sgn} \in \widehat{\mathbb{D}_n^x}$ be the sign representation ($\mathbb{D}_n^x = \langle x \rangle \simeq \mathbb{Z}_2$). The goal of this Section is to apply the next result, cf. [AHS, Th. 4.8], or [AHS, Th. 4.5] for $n = 3$.

Theorem 2.1. *The Nichols algebra $\mathfrak{B}(M(\mathcal{O}_x, \operatorname{sgn}) \oplus M(\mathcal{O}_x, \operatorname{sgn}))$ has infinite dimension.* □

Note that $M(\mathcal{O}_x, \operatorname{sgn}) \oplus M(\mathcal{O}_x, \operatorname{sgn})$ is isomorphic as a braided vector space to $(\mathbb{C}X_n, \mathfrak{q})$, where

- X_n is the rack with $2n$ elements s_i, t_j, $i, j \in \mathbb{Z}/n$, and with structure

$$s_i \triangleright s_j = s_{2i-j}, \quad s_i \triangleright t_j = t_{2i-j}, \quad t_i \triangleright s_j = s_{2i-j}, \quad t_i \triangleright t_j = t_{2i-j}, \quad i, j \in \mathbb{Z}/n;$$

- $\mathfrak{q}$ is the constant cocycle $\mathfrak{q} \equiv -1$.

If d divides n, then X_d can be identified with a subrack of X_n. Hence, it is enough to consider braided vector spaces $(\mathbb{C}X_p, \mathfrak{q})$, with p an odd prime.

We fix a finite group G with the rack structure given by conjugation $x \triangleright y = xyx^{-1}$, x, $y \in G$. Let $\mathcal{O}$ be a conjugacy class in G.

Definition 2.2. Let $p > 1$ be an integer. A family $(\mu_i)_{i\in\mathbb{Z}/p}$ of distinct elements of G is *of type* $\mathcal{D}_p$ if

$$\mu_i \triangleright \mu_j = \mu_{2i-j}, \quad i, j \in \mathbb{Z}/p. \tag{2.1}$$

Let $(\mu_i)_{i\in\mathbb{Z}/p}$ and $(\nu_i)_{i\in\mathbb{Z}/p}$ be two families of type $\mathcal{D}_p$ in G, such that $\mu_i \neq \nu_j$ for all $i, j \in \mathbb{Z}/p$. Then $(\mu, \nu) := (\mu_i)_{i\in\mathbb{Z}/p} \cup (\nu_i)_{i\in\mathbb{Z}/p}$ is *of type* $\mathcal{D}_p^{(2)}$ if

$$\mu_i \triangleright \nu_j = \nu_{2i-j}, \quad \nu_i \triangleright \mu_j = \mu_{2i-j}, \quad i, j \in \mathbb{Z}/p. \tag{2.2}$$

It is useful to denote $i \triangleright j = 2i - j$, for $i, j \in \mathbb{Z}/p$.

We state some consequences of this definition for further use.

Remark 2.3. If $(\mu_i)_{i\in\mathbb{Z}/p}$ is of type $\mathcal{D}_p$ then

$$\mu_i^{-1} \triangleright \mu_j = \mu_{2i-j}, \qquad \mu_i \triangleright \mu_j^{-1} = \mu_{2i-j}^{-1}, \qquad \mu_i^{-1} \triangleright \mu_j^{-1} = \mu_{2i-j}^{-1}, \tag{2.3}$$

$$\mu_i^{k} \triangleright \mu_j = \mu_{2i-j}, \qquad \mu_i \triangleright \mu_j^{k} = \mu_{2i-j}^{k}, \qquad \mu_i^{k} \triangleright \mu_j^{k} = \mu_{2i-j}^{k}, \tag{2.4}$$

for all $i, j \in \mathbb{Z}/p$, and for all k odd.

Remark 2.4. Assume that p is odd. If $(\mu, \nu) = (\mu_i)_{i\in\mathbb{Z}/p} \cup (\nu_i)_{i\in\mathbb{Z}/p}$ is of type $\mathcal{D}_p^{(2)}$, then for all i, j,

$$\mu_i^2 = \mu_j^2, \quad \nu_i^2 = \nu_j^2, \quad \mu_i^2 \nu_j = \nu_j \mu_i^2, \quad \nu_i^2 \mu_j = \mu_j \nu_i^2. \tag{2.5}$$

Indeed, $\mu_h^2 \mu_j = \mu_j \mu_h^2$, hence $\mu_{2h-j}^2 = \mu_h \mu_j^2 \mu_h^{-1} = \mu_j^2$. Take now $h = \dfrac{i+j}{2}$.

Lemma 2.5. *If* $(\mu, \nu) = (\mu_i)_{i\in\mathbb{Z}/p} \cup (\nu_i)_{i\in\mathbb{Z}/p}$ *is of type* $\mathcal{D}_p^{(2)}$*, then*

(i) $\mu_k\mu_l = \mu_{t(l-k)+k}\ \mu_{t(l-k)+l}$,

(ii) $\mu_k\nu_l = \mu_{2t(l-k)+k}\ \nu_{2t(l-k)+l}$,

(iii) $\mu_k\nu_l = \nu_{(2t+1)(l-k)+k}\ \mu_{(2t+1)(l-k)+l}$,

for all $k,\ l,\ t \in \mathbb{Z}/p$.

Notice that we have the analogous relations interchanging μ by ν.

Proof. We proceed by induction on t. We will prove (i); (ii) and (iii) are similar. The result is obvious when $t = 0$. Since $\mu_k\mu_l = \mu_l\ \mu_{l\triangleright k}$, then the result holds for $t = 1$. Let us suppose that (i) holds for every $s \le t$. Now,

$$\begin{aligned}\mu_k\mu_l &= \mu_{t(l-k)+k}\ \mu_{t(l-k)+l}\\ &= \mu_{t(l-k)+l}\ \mu_{(t(l-k)+l)\triangleright(t(l-k)+k)} = \mu_{(t+1)(l-k)+k}\ \mu_{(t+1)(l-k)+l}\end{aligned}$$

by the recursive hypothesis. □

Lemma 2.6. *Assume that* p *is odd. If* (μ, ν) *is of type* $\mathcal{D}_p^{(2)}$*, then for* $i \in \mathbb{Z}/p$*,*

$$\mu_i\nu_i = \mu_0\nu_0, \tag{2.6}$$

$$\nu_i\mu_i = \nu_0\mu_0. \tag{2.7}$$

Proof. Let $i,\ j \in \mathbb{Z}/p$, with $i \neq j$. If we write (ii) of Lemma 2.5 with $k = i$, $l = j$ and $t = -1/2$ we have that $\mu_i\nu_j = \mu_{2i-j}\nu_i$. Thus, $\mu_i\nu_i\nu_j^2 = \mu_i\nu_j\nu_j\nu_i = \mu_{2i-j}\nu_i\nu_i\nu_{2i-j} = \mu_{2i-j}\nu_{2i-j}\nu_i^2$, and, by (2.5),

$$\mu_i\nu_i = \mu_{2i-j}\nu_{2i-j}.$$

Now (2.6) follows taking $j = 2i$. Now (2.7) follows from (2.6) by (2.2). □

We now set up some notation that will be used in the rest of this section. Let $(\mu_i)_{i\in\mathbb{Z}/p}$ be a family of type $\mathcal{D}_p$ in G, with p odd. Set

$$g_i = \mu_{i/2}, \tag{2.8}$$

$$\alpha_{ij} = g_{i\triangleright j}^{-1}\,\mu_i\,g_j = \mu_{i-j/2}^{-1}\,\mu_i\,\mu_{j/2}, \tag{2.9}$$

for all $i, j \in \mathbb{Z}/p$. Then

$$g_i \triangleright \mu_0 = \mu_i, \qquad \alpha_{ij} \in G^{\mu_0}, \qquad i, j \in \mathbb{Z}/p.$$

Let now (μ, ν) be of type $\mathcal{D}_p^{(2)}$ and suppose that there exists $g_\infty \in G$ such that $g_\infty \triangleright \mu_0 = \nu_0$. Set

$$f_i = \nu_{i/2}\,g_\infty, \tag{2.10}$$

$$\beta_{ij} = f_{i\triangleright j}^{-1}\,\mu_i\,f_j = g_\infty^{-1}\,\nu_{i-j/2}^{-1}\,\mu_i\,\nu_{j/2}\,g_\infty, \tag{2.11}$$

$$\gamma_{ij} = g_{i\triangleright j}^{-1}\,\nu_i\,g_j = \mu_{i-j/2}^{-1}\,\nu_i\,\mu_{j/2}, \tag{2.12}$$

$$\delta_{ij} = f_{i\triangleright j}^{-1}\,\nu_i\,f_j = g_\infty^{-1}\,\nu_{i-j/2}^{-1}\,\nu_i\,\nu_{j/2}\,g_\infty. \tag{2.13}$$

Then

$$f_i \triangleright \mu_0 = \nu_i, \qquad \beta_{ij}, \gamma_{ij}, \delta_{ij} \in G^{\mu_0}, \qquad i, j \in \mathbb{Z}/p.$$

We assume from now on that p is an odd prime. This is required in the proof of the next lemma, needed for the main result of the section.

Lemma 2.7. *Let $(\mu, \nu) = (\mu_i)_{i\in\mathbb{Z}/p} \cup (\nu_i)_{i\in\mathbb{Z}/p}$ be of type $\mathcal{D}_p^{(2)}$, and suppose that there exists $g_\infty \in G$ such that $g_\infty \triangleright \mu_0 = \nu_0$. Let g_i and f_i be as in* (2.8) *and* (2.10)*, respectively. Then, for all $i, j \in \mathbb{Z}/p$,*

(a) $\alpha_{ij} = \delta_{ij} = \mu_0$,
(b) $\beta_{ij} = g_\infty^{-1}\mu_0 g_\infty$,
(c) $\gamma_{ij} = \nu_0$.

Proof. Let k, l be in $\mathbb{Z}/p$. Then, for all $r \in \mathbb{Z}/p$, we have

$$\mu_k\mu_l = \mu_{k+r}\mu_{l+r}, \quad \mu_k\nu_l = \mu_{k+r}\nu_{l+r}, \quad \mu_k\nu_l = \nu_{k+r}\mu_{l+r}. \tag{2.14}$$

This follows from (2.5) and Lemma 2.6 (when $k = l$), and Lemma 2.5 (when $k \neq l$). There are similar equalities interchanging μ's and ν's. Now

$$\begin{aligned}
\alpha_{ij} &= \mu_{i-j/2}^{-1}\,\mu_i\,\mu_{j/2} \overset{(2.14)}{=} \mu_0,\\
\delta_{ij} &= g_\infty^{-1}\nu_{i-j/2}^{-1}\,\nu_i\,\nu_{j/2}\,g_\infty \overset{(2.14)}{=} g_\infty^{-1}\nu_0\,g_\infty = \mu_0,\\
\beta_{ij} &= g_\infty^{-1}\nu_{i-j/2}^{-1}\,\mu_i\,\nu_{j/2}\,g_\infty \overset{(2.14)}{=} g_\infty^{-1}\,\mu_0\,g_\infty,\\
\gamma_{ij} &= \mu_{i-j/2}^{-1}\,\nu_i\,\mu_{j/2} \overset{(2.14)}{=} \mu_{i-j/2}^{-1}\,\mu_{i-j/2}\nu_0 = \nu_0,
\end{aligned}$$

and the Lemma is proved. $\square$

We can now prove one of the main results of this paper.

Theorem 2.8. *Let $(\mu, \nu) = (\mu_i)_{i\in\mathbb{Z}/p} \cup (\nu_i)_{i\in\mathbb{Z}/p}$ be a family of elements in G with $\mu_0 \in \mathcal{O}$. Let (ρ, V) be an irreducible representation of the centralizer G^{μ_0}. We assume that*

(H1) *(μ, ν) is of type $\mathcal{D}_p^{(2)}$;*
(H2) *$(\mu, \nu) \subseteq \mathcal{O}$, with $g_\infty \in G$ such that $g_\infty \triangleright \mu_0 = \nu_0$;*
(H3) *$q_{\mu_0\mu_0} = -1$;*
(H4) *there exist $v, w \in V - 0$ such that,*

$$\rho(g_\infty^{-1}\mu_0 g_\infty)w = -w, \tag{2.15}$$

$$\rho(\nu_0)v = -v. \tag{2.16}$$

Then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Proof. We keep the notation (2.10)–(2.13) above. Let $v, w \in V - 0$ as in (H4) and let $W :=$ span-$\{g_i v : i \in \mathbb{Z}/p\} \cup \{f_i w : i \in \mathbb{Z}/p\}$. Let $\Psi : \mathbb{C}X_p \to W$ be given by $\Psi(s_i) = g_i v$, $\Psi(t_i) = f_i w$, $i \in \mathbb{Z}/p$. Since the elements μ_i and ν_j are all different, Ψ is a linear isomorphism. We claim that W is a braided vector subspace of $M(\mathcal{O}, \rho)$ and that Ψ is an isomorphism of braided vector spaces. We compute the braiding in W:

$$c(g_i v \otimes g_j v) = \mu_i g_j v \otimes g_i v = g_{i \triangleright j} \alpha_{ij} v \otimes g_i v \overset{(\text{H3})}{=} -g_{i \triangleright j} v \otimes g_i v,$$
$$c(g_i v \otimes f_j w) = \mu_i f_j w \otimes g_i v = f_{i \triangleright j} \beta_{ij} w \otimes g_i v \overset{(2.15)}{=} -f_{i \triangleright j} w \otimes g_i v,$$
$$c(f_i w \otimes g_j v) = \nu_i g_j v \otimes f_i w = g_{i \triangleright j} \gamma_{ij} v \otimes f_i w \overset{(2.16)}{=} -g_{i \triangleright j} v \otimes f_i w,$$
$$c(f_i w \otimes f_j w) = \nu_i f_j w \otimes f_i w = f_{i \triangleright j} \delta_{ij} w \otimes f_i w \overset{(\text{H3})}{=} -f_{i \triangleright j} w \otimes f_i w,$$

by Lemma 2.7. The claim is proved. Hence, $\dim \mathfrak{B}(W) = \infty$ by Theorem 2.1. Now the Theorem follows from Lemma 1.2. $\square$

As a consequence of Theorem 2.8, we can state a very useful criterion.

Corollary 2.9. *Let G be a finite group, μ_i, $0 \leq i \leq p-1$, distinct elements in G, with p an odd prime. Let us suppose that there exists $k \in \mathbb{Z}$ such that $\mu_0^k \neq \mu_0$ and $\mu_0^k \in \mathcal{O}$, the conjugacy class of μ_0. Let $\rho = (\rho, V) \in \widehat{G^{\mu_0}}$. Assume further that*

(i) $(\mu_i)_{i \in \mathbb{Z}/p}$ *is of type* $\mathcal{D}_p$,

(ii) $q_{\mu_0 \mu_0} = -1$.

Then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Proof. We may assume that $1 < k < |\mu_0|$. By hypothesis (ii), the order of μ_0 is even; hence k is odd, say $k = 2t+1$, with $t \geq 1$. Let $\nu_i := \mu_i^k$, $0 \leq i \leq p-1$, and pick $g_\infty \in G$ such that $g_\infty \triangleright \mu_0 = \mu_0^k$. Set $(\mu, \nu) = (\mu_i)_{i \in \mathbb{Z}/p} \cup (\nu_i)_{i \in \mathbb{Z}/p}$. Clearly $(\mu, \nu) \subseteq \mathcal{O}$. We claim that (μ, ν) is of type $\mathcal{D}_p^{(2)}$. Indeed, using (i) it is easy to see that $(\mu_i)_{i \in \mathbb{Z}/p} \cup (\nu_i)_{i \in \mathbb{Z}/p}$ are all distinct. Then the claim follows by (2.4).

It remains to check the hypothesis (H4) of Theorem 2.8. As $g_\infty \mu_0 g_\infty^{-1} = \mu_0^k$, $g_\infty^l \mu_0 g_\infty^{-l} = \mu_0^{k^l}$, for all $l \geq 0$. In particular,

$$g_\infty^{-1} \mu_0 g_\infty = g_\infty^{|g_\infty|-1} \mu_0 g_\infty^{-|g_\infty|+1} = \mu_0^{k^{|g_\infty|-1}}.$$

Then, since $q_{\mu_0 \mu_0} = -1$ and k is odd, we see that $\rho(g_\infty^{-1} \mu_0 g_\infty) = -\operatorname{id}$. Hence (2.15) holds, for any $w \in V - 0$. Also, $\rho(\nu_0) = \rho(\mu_0^k) = (-\operatorname{id})^k = -\operatorname{id}$, because k is odd; thus, (2.16) holds for any $v \in V - 0$. Thus, for any v, w in $V - 0$, we are in the conditions of Theorem 2.8. Then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$. $\square$

Example 2.10. *Let $m \geq 6$. Let $\sigma \in \mathbb{S}_m$ of type $(1^{n_1}, 2^{n_2}, \dots, m^{n_m})$, $\mathcal{O}$ the conjugacy class of σ and $\rho \in \widehat{\mathbb{S}_m^\sigma}$. If there exists j, $1 \leq j \leq m$, such that*

- *$2p$ divides j, for some odd prime p, and*
- *$n_j \geq 1$;*

then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Before proving the Example, we state a more general Lemma that might be of independent interest. Here p is no longer an odd prime.

Lemma 2.11. *Let $m, p \in \mathbb{Z}_{>1}$. Let $\sigma \in \mathbb{S}_m$ of type $(1^{n_1}, 2^{n_2}, \dots, m^{n_m})$ and $\mathcal{O}$ the conjugacy class of σ. If there exists $j \neq 4$, $1 \leq j \leq m$, such that*

- *$2p$ divides j, and*
- *$n_j \geq 1$;*

then $\mathcal{O}$ contains a subrack of type $\mathcal{D}_p^{(2)}$.

Proof. Let $j = 2p\,\kappa$, with $\kappa \geq 1$. Let $\alpha = (i_1\, i_2\, \cdots\, i_j)$ be a j-cycle that appears in the decomposition of σ as product of disjoint cycles and define

$$\mathbf{I} := (i_1\, i_3\, i_5\, \cdots\, i_{j-1}) \quad \text{and} \quad \mathbf{P} := (i_2\, i_4\, i_6\, \cdots\, i_j).$$

We claim that

(a) $\mathbf{I}$ and $\mathbf{P}$ are disjoint $p\kappa$-cycles,
(b) $\alpha^2 = \mathbf{IP}$,
(c) $\alpha \mathbf{I} \alpha^{-1} = \mathbf{P}$, (and then $\sigma \mathbf{I} \sigma^{-1} = \mathbf{P}$),
(d) $\mathbf{P}^t \alpha \mathbf{P}^t = \alpha^{2t+1}$, $\mathbf{P}^t \alpha^{-1} \mathbf{P}^t = \alpha^{2t-1}$, for all integers t.

The first two items are clear, while (c) follows from the well-known formula $\alpha(l_1\, l_2\, \dots\, l_k)\alpha^{-1} = (\alpha(l_1)\, \alpha(l_2)\, \dots\, \alpha(l_k))$. (d). By (c), $\mathbf{P}^t = \alpha \mathbf{I}^t \alpha^{-1}$. Then $\mathbf{P}^t \alpha \mathbf{P}^t = \alpha \mathbf{I}^t \mathbf{P}^t$; by (b), $\mathbf{P}^t \alpha \mathbf{P}^t = \alpha\alpha^{2t}$, as claimed.

We define

$$\sigma_i := \mathbf{P}^{i\kappa} \sigma \mathbf{P}^{-i\kappa}, \qquad 0 \leq i \leq p-1. \tag{2.17}$$

Notice that $\sigma_i = \mathbf{P}^{i\kappa} \alpha \mathbf{P}^{-i\kappa}\, \widetilde{\sigma}$, where $\widetilde{\sigma} := \alpha^{-1}\sigma$. The elements $(\sigma_i)_{i \in \mathbb{Z}/p}$ are all distinct; indeed, if $\sigma_i = \sigma_l$, with i, $l \in \mathbb{Z}/p$, then $\mathbf{P}^{i\kappa}\sigma\mathbf{P}^{-i\kappa} = \mathbf{P}^{l\kappa}\sigma\mathbf{P}^{-l\kappa}$, i. e. $\mathbf{P}^{(i-l)\kappa}\sigma\mathbf{P}^{-(i-l)\kappa} = \sigma$, which implies that $i_2 = \sigma(i_1) = \mathbf{P}^{(i-l)\kappa}\sigma\mathbf{P}^{-(i-l)\kappa}(i_1) = \mathbf{P}^{(i-l)\kappa}(i_2) = i_{2(i-l)\kappa+2}$, and this means that $2(i-l)\kappa = 0$ in $\mathbb{Z}/j$. Thus $i = l$, as desired.

We claim that $(\sigma_i)_{i\in\mathbb{Z}/p}$ is of type $\mathcal{D}_p$. If i, $l \in \mathbb{Z}/p$, then

$$\begin{aligned}
\sigma_i \triangleright \sigma_l &= \mathbf{P}^{i\kappa}\sigma\mathbf{P}^{-i\kappa}\ \mathbf{P}^{l\kappa}\sigma\mathbf{P}^{-l\kappa}\ \mathbf{P}^{i\kappa}\sigma^{-1}\mathbf{P}^{-i\kappa} \\
&= \mathbf{P}^{i\kappa}\,\alpha\,\mathbf{P}^{-i\kappa}\ \mathbf{P}^{l\kappa}\,\alpha\,\mathbf{P}^{-l\kappa}\ \mathbf{P}^{i\kappa}\alpha^{-1}\mathbf{P}^{-i\kappa}\ \widetilde{\sigma} \\
&= \mathbf{P}^{(2i-l)\kappa}\,\mathbf{P}^{(l-i)\kappa}\,\alpha\,\mathbf{P}^{(l-i)\kappa}\,\alpha\,\mathbf{P}^{(i-l)\kappa}\,\alpha^{-1}\,\mathbf{P}^{(i-l)\kappa}\,\mathbf{P}^{-(2i-l)\kappa}\ \widetilde{\sigma} \\
&= \mathbf{P}^{(2i-l)\kappa}\,\alpha^{2(l-i)\kappa+1}\,\alpha\,\alpha^{2(i-l)\kappa-1}\,\mathbf{P}^{-(2i-l)\kappa}\ \widetilde{\sigma} \\
&= \mathbf{P}^{(2i-l)\kappa}\,\alpha\,\mathbf{P}^{-(2i-l)\kappa}\ \widetilde{\sigma} = \mathbf{P}^{(2i-l)\kappa}\,\sigma\,\mathbf{P}^{-(2i-l)\kappa} = \sigma_{i\triangleright l},
\end{aligned}$$

by (d), and the claim follows. Finally, the family of type $\mathcal{D}_p^{(2)}$ we are looking for is $(\sigma_i)_{i\in\mathbb{Z}/p}\cup(\sigma_i^{-1})_{i\in\mathbb{Z}/p}$. It remains to show that $\sigma_t \neq \sigma_l^{-1}$ for all t, $l \in \mathbb{Z}_p$. If $\sigma_t = \sigma_l^{-1}$, then $\sigma_t^2(i_1) = \sigma_l^{-2}(i_1)$, that is $i_3 = i_{j-1}$, a contradiction to the hypothesis $j \neq 4$. □

Proof of the Example 2.10. We may assume that $q_{\sigma\sigma} = -1$, by Lemma 1.3. By Lemma 2.11, we have a family $(\sigma_i)_{i\in\mathbb{Z}/p}$ of type $\mathcal{D}_p$, with $\sigma_0 = \sigma$. Now Corollary 2.9 applies, with $\mu_0 = \sigma_0$, $k = |\sigma_0|-1$. Thus $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$. □

3. A technique from the symmetric group $\mathbb{S}_3$

We study separately the case $p = 3$ because of the many applications found. In this setting, $\mathbb{D}_3 \simeq \mathbb{S}_3$ and $\mathcal{O}_x = \mathcal{O}_2^3 = \{(1\,2), (2\,3), (1\,3)\}$ is the conjugacy class of transpositions in $\mathbb{S}_3$. The rack X_3 is described as a set of 6 elements $X_3 = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ with the multiplication

$$x_i \triangleright x_j = x_k, \quad y_i \triangleright y_j = y_k, \quad x_i \triangleright y_j = y_k, \quad y_i \triangleright x_j = x_k,$$

for i, j, k, all distinct or all equal.

3.1. Families of type $\mathcal{D}_3$ and $\mathcal{D}_3^{(2)}$.

We fix a finite group G and $\mathcal{O}$ a conjugacy class in G. Our aim is to give criteria to detect when $\mathcal{O}$ contains a subrack isomorphic to X_3.

Definition 3.1. Let σ_1, σ_2, $\sigma_3 \in G$ distinct. We say that $(\sigma_i)_{1\leq i\leq 3}$ is of type $\mathcal{D}_3$ if

$$\sigma_i \triangleright \sigma_j = \sigma_k, \qquad \text{where } i,\ j,\ k \text{ are all distinct.} \tag{3.1}$$

The requirement (3.1) consists of 6 identities, but actually 3 are enough.

Remark 3.2. If

$$\sigma_1 \triangleright \sigma_2 = \sigma_3, \tag{3.2}$$

$$\sigma_1 \triangleright \sigma_3 = \sigma_2, \tag{3.3}$$

$$\sigma_2 \triangleright \sigma_3 = \sigma_1, \tag{3.4}$$

then $(\sigma_i)_{1\leq i\leq 3}$ is of type $\mathcal{D}_3$. □

Here is a characterization of $\mathcal{D}_3$ families.

Proposition 3.3. *Let* σ_1, $\sigma_2 \in \mathcal{O}$. *Define* $\sigma_3 := \sigma_1 \triangleright \sigma_2$. *Then* $(\sigma_i)_{1\le i\le 3}$ *is of type* $\mathcal{D}_3$ *if and only if*

$$\sigma_1 \notin G^{\sigma_2}, \tag{3.5}$$

$$\sigma_1^2 \in G^{\sigma_2}, \tag{3.6}$$

$$\sigma_1 = \sigma_2 \triangleright (\sigma_1 \triangleright \sigma_2). \tag{3.7}$$

Proof. The definition of σ_3 is equivalent to (3.2) and (3.7) is equivalent to (3.4). Assume that $(\sigma_i)_{1\le i\le 3}$ is of type $\mathcal{D}_3$. As $\sigma_3 \neq \sigma_2$, $\sigma_1 \notin G^{\sigma_2}$. Also, $\sigma_1^2 \triangleright \sigma_2 = \sigma_1 \triangleright (\sigma_1 \triangleright \sigma_2) = \sigma_1 \triangleright \sigma_3 = \sigma_2$. Hence $\sigma_1^2 \in G^{\sigma_2}$.

Conversely, if $\sigma_1 \notin G^{\sigma_2}$, then $\sigma_1 \neq \sigma_2$, $\sigma_2 \neq \sigma_3$. From (3.5) and (3.7), we see that $\sigma_1 \neq \sigma_3$. It remains to check (3.3): $\sigma_1 \triangleright \sigma_3 = \sigma_1^2 \triangleright \sigma_2 = \sigma_2$. $\square$

Definition 3.4. Let σ_1, σ_2, σ_3, τ_1, τ_2, $\tau_3 \in G$ be distinct elements. We say that $(\sigma, \tau) = (\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3)$ is of type $\mathcal{D}_3^{(2)}$, if $(\sigma_i)_{1\le i\le 3}$ and $(\tau_j)_{1\le j\le 3}$ are of type $\mathcal{D}_3$, and

$$\sigma_i \triangleright \tau_j = \tau_k, \qquad \tau_i \triangleright \sigma_j = \sigma_k, \tag{3.8}$$

where i, j, k are either all equal, or all distinct.

The requirement (3.8) consists of 18 identities, but less are enough. We begin by a first reduction.

Lemma 3.5. *Let* $(\sigma_i)_{1\le i\le 3}$ *and* $(\tau_j)_{1\le j\le 3}$ *such that* (3.2), (3.3), (3.4) *hold for* σ *and for* τ. *If*

$$\sigma_1 \triangleright \tau_1 = \tau_1, \tag{3.9}$$

$$\sigma_1 \triangleright \tau_2 = \tau_3, \tag{3.10}$$

$$\sigma_2 \triangleright \tau_1 = \tau_3, \tag{3.11}$$

also hold, then $\sigma_i \triangleright \tau_i = \tau_i$, $1 \le i \le 3$, *and* $\sigma_i \triangleright \tau_j = \tau_k$, *for all* i, j, k *distinct.*

Proof. We have to prove

$$\sigma_1 \triangleright \tau_3 = \tau_2, \tag{3.12}$$

$$\sigma_3 \triangleright \tau_3 = \tau_3, \tag{3.13}$$

$$\sigma_2 \triangleright \tau_2 = \tau_2, \tag{3.14}$$

$$\sigma_3 \triangleright \tau_1 = \tau_2, \tag{3.15}$$

$$\sigma_3 \triangleright \tau_2 = \tau_1, \tag{3.16}$$

$$\sigma_2 \triangleright \tau_3 = \tau_1, \tag{3.17}$$

The identity (3.12) holds because $\sigma_1 \triangleright \tau_3 = \sigma_1 \triangleright (\tau_1 \triangleright \tau_2) = \tau_1 \triangleright \tau_3 = \tau_2$; in turn, (3.13) and (3.14) hold because

$$\sigma_3 \triangleright \tau_3 = (\sigma_2 \triangleright \sigma_1) \triangleright (\sigma_2 \triangleright \tau_1) = \sigma_2 \triangleright (\sigma_1 \triangleright \tau_1) = \sigma_2 \triangleright \tau_1 = \tau_3,$$
$$\sigma_2 \triangleright \tau_2 = (\sigma_1 \triangleright \sigma_3) \triangleright (\sigma_1 \triangleright \tau_3) = \sigma_1 \triangleright (\sigma_3 \triangleright \tau_3) = \sigma_1 \triangleright \tau_3 = \tau_2.$$

Also, $\sigma_3 \triangleright \tau_1 = (\sigma_1 \triangleright \sigma_2) \triangleright (\sigma_1 \triangleright \tau_1) = \sigma_1 \triangleright (\sigma_2 \triangleright \tau_1) = \sigma_1 \triangleright \tau_3 = \tau_2$, showing (3.15). Finally, $\sigma_3 \triangleright \tau_2 = \sigma_3 \triangleright (\sigma_1 \triangleright \tau_3) = \sigma_2 \triangleright (\sigma_3 \triangleright \tau_3) = \sigma_2 \triangleright \tau_3 = \sigma_2 \triangleright (\tau_1 \triangleright \tau_2) = \tau_3 \triangleright \tau_2 = \tau_1$, proving (3.16) and (3.17). $\square$

Therefore, given 6 distinct elements σ_1, σ_2, σ_3, τ_1, τ_2, $\tau_3 \in G$, if the 12 identities: (3.2), (3.3), (3.4), for σ and for τ, (3.9), (3.10), (3.11), and the analogous identities

$$\tau_1 \triangleright \sigma_1 = \sigma_1, \tag{3.18}$$
$$\tau_1 \triangleright \sigma_2 = \sigma_3, \tag{3.19}$$
$$\tau_2 \triangleright \sigma_1 = \sigma_3, \tag{3.20}$$

hold, then (σ, τ) is of type $\mathcal{D}_3^{(2)}$. But we can get rid of 3 of these 12 identities.

Proposition 3.6. *Let σ_1, σ_2, σ_3, τ_1, τ_2, $\tau_3 \in G$, all distinct, such that* (3.2), (3.3), (3.4), *hold for σ and for τ, as well as the identities* (3.9), (3.11) *and* (3.19). *Then (σ, τ) is of type $\mathcal{D}_3^{(2)}$.*

Proof. By Lemma 3.5, it is enough to check (3.10), (3.18) and (3.20). First, (3.18) holds because $\tau_1 = \sigma_1 \triangleright \tau_1 = \sigma_1 \tau_1 \sigma_1^{-1}$. If τ_1 acts on both sides of (3.11), then $\tau_2 = \tau_1 \triangleright \tau_3 = (\tau_1 \triangleright \sigma_2) \triangleright (\tau_1 \triangleright \tau_1) = \sigma_3 \triangleright \tau_1$; if now σ_1 acts on the last, then

$$\sigma_1 \triangleright \tau_2 = (\sigma_1 \triangleright \sigma_3) \triangleright (\sigma_1 \triangleright \tau_1) = \sigma_2 \triangleright \tau_1 \overset{(3.11)}{=} \tau_3.$$

Thus, (3.10) holds. We can now conclude from Lemma 3.5 that $\sigma_i \triangleright \tau_i = \tau_i$, $1 \le i \le 3$, and $\sigma_i \triangleright \tau_j = \tau_k$, for all i, j, k distinct. If now σ_3 acts on (3.19), then $\sigma_3 = (\sigma_3 \triangleright \tau_1) \triangleright (\sigma_3 \triangleright \sigma_2) = \tau_2 \triangleright \sigma_1$, and (3.20) holds. $\square$

3.2. **Examples of $\mathcal{D}_3^{(2)}$ type.** We first spell out explicitly Theorem 2.8 and Corollary 2.9 for $p = 3$.

Theorem 3.7. *Let σ_1, σ_2, σ_3, τ_1, τ_2, $\tau_3 \in G$ distinct; denote $(\sigma, \tau) = (\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3)$. Let $\rho = (\rho, V) \in \widehat{G^{\sigma_1}}$. We assume that*

(H1) *(σ, τ) is of type $\mathcal{D}_3^{(2)}$,*
(H2) *$(\sigma, \tau) \subseteq \mathcal{O}$, with $g \in G$ such that $g \triangleright \sigma_1 = \tau_1$,*
(H3) *$q_{\sigma_1 \sigma_1} = -1$,*

(H4) *there exist* $v, w \in V - 0$ *such that,*

$$\rho(g^{-1}\sigma_1 g)w = -w, \tag{3.21}$$

$$\rho(\tau_1)v = -v, \tag{3.22}$$

Then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$. □

Corollary 3.8. *Let* σ_1, σ_2, $\sigma_3 \in \mathcal{O}$ *distinct. Assume that there exists* k, $1 \leq k \leq |\sigma_1|$, *such that* $\sigma_1^k \neq \sigma_1$ *and* $\sigma_1^k \in \mathcal{O}$. *Let* $\rho = (\rho, V) \in \widehat{G^{\sigma_1}}$. *Assume further that*

(1) $(\sigma_i)_{1\leq i\leq 3}$ *is of type* $\mathcal{D}_3$,
(2) $q_{\sigma_1\sigma_1} = -1$.

Then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$. □

Corollary 3.8 applies notably to a real conjugacy class of an element of order greater than 2. We list several applications for $G = \mathbb{S}_m$.

Example 3.9. *Let* $m \geq 6$. *Let* $\mathcal{O}$ *be the conjugacy class of* $\mathbb{S}_m$ *of type* $(1^{n_1}, 2^{n_2}, \dots, m^{n_m})$, *where*

- n_1, $n_2 \geq 1$ *and*
- $n_j \geq 1$ *for some* j, $3 \leq j \leq m$.

Let $\sigma \in \mathcal{O}$ *and* $\rho \in \widehat{\mathbb{S}_m^\sigma}$. *Then* $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Proof. By hypothesis, we can choose $\sigma = (1\,2)\beta$ where β fixes 1, 2 and 3. If $q_{\sigma\sigma} \neq -1$, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$, by Lemma 1.3. Assume that $q_{\sigma\sigma} = -1$. Now set

$$x = (1\,2), \quad y = (1\,3), \quad z = (2\,3), \quad \sigma_1 = \sigma = x\beta, \quad \sigma_2 = y\beta, \quad \sigma_3 := z\beta.$$

Clearly $(\sigma_i)_{1\leq i\leq 3}$ is of type $\mathcal{D}_3$, $\mathcal{O}$ is real and $|\sigma_1| > 2$. By Corollary 3.8, $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$. □

In particular, let $\mathcal{O}$ be the conjugacy class of $\mathbb{S}_m$ of type $(1, 2, m-3)$, with $m \geq 6$. By the preceding, $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$. But, if $q_{\sigma\sigma} = -1$, then $M(\mathcal{O}, \rho)$ has negative braiding; that is, it is not possible to decide if the dimension of $\mathfrak{B}(\mathcal{O}, \rho)$ is infinite via abelian subracks. See [F2] for details.

Example 3.10. *Let* $m \geq 6$. *Let* $\sigma \in \mathbb{S}_m$ *of type* $(1^{n_1}, 2^{n_2}, \dots, m^{n_m})$, $\mathcal{O}$ *the conjugacy class of* σ *and* $\rho \in \widehat{\mathbb{S}_m^\sigma}$. *Assume that*

- *there exists* j, $1 \leq j \leq m$, *such that* $j = 2k$, *with* $k \geq 2$ *and* $n_j \geq 3$.

Then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Proof. If $q_{\sigma\sigma} \neq -1$, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$, by Lemma 1.3. Assume that $q_{\sigma\sigma} = -1$. Let

$$\alpha_1 = (i_1\, i_2\, \cdots\, i_j), \quad \alpha_2 = (i_{j+1}\, i_{j+2}\, \cdots\, i_{2j}), \quad \alpha_3 = (i_{2j+1}\, i_{2j+2}\, \cdots\, i_{3j}),$$

be three j-cycles appearing in the decomposition of σ as product of disjoint cycles and define

$$\begin{aligned} \mathbf{I} &= (i_1\, i_3\, i_5\, \cdots\, i_{3j-1}), \qquad & B_1 &= (i_1\ i_{j+1})(i_2\ i_{j+2})\cdots(i_j\ i_{2j}),\\ \mathbf{P} &= (i_2\, i_4\, i_6\, \cdots\, i_{3j}), & B_2 &= (i_{j+1}\ i_{2j+1})(i_{j+2}\ i_{2j+2})\cdots(i_{2j}\ i_{3j}). \end{aligned}$$

Then

(a) $\mathbf{I}$ and $\mathbf{P}$ are disjoint $3k$-cycles,
(b) $\mathbf{I}^k\mathbf{P}^k = B_1B_2$,
(c) $\alpha_1\alpha_2\alpha_3\mathbf{I}\alpha_3^{-1}\alpha_2^{-1}\alpha_1^{-1} = \mathbf{P}$, (and then $\sigma\mathbf{I}\sigma^{-1} = \mathbf{P}$),
(d) $\mathbf{P}^k\sigma\mathbf{P}^k = \sigma B_1B_2$, and
(e) $\mathbf{P}^{-k}\sigma\mathbf{P}^{-k} = \sigma B_2B_1$.

The first item is clear. To see (b), note that

$$B_1B_2 = (i_1\ i_{j+1}\ i_{2j+1})(i_2\ i_{j+2}\ i_{2j+2})\cdots(i_j\ i_{2j}\ i_{3j}).$$

(c) follows as in the proof of Lemma 2.11 (c). (d). By (b) and (c), we have that $\sigma^{-1}\mathbf{P}^k\sigma\mathbf{P}^k = \mathbf{I}^k\mathbf{P}^k = B_1B_2$, as claimed. (e). By (b) and (c), $\sigma^{-1}\mathbf{P}^{-k}\sigma\mathbf{P}^{-k} = \mathbf{I}^{-k}\mathbf{P}^{-k} = B_2B_1$ as claimed.

Set now $\sigma_1 := \sigma$, $\sigma_2 := \mathbf{P}^k\sigma\mathbf{P}^{-k}$ and $\sigma_3 := \mathbf{P}^{-k}\sigma\mathbf{P}^k$. As in the proof of Example 2.10 we can see that σ_1, σ_2 and σ_3 are distinct. We check that $(\sigma_i)_{1\leq i\leq 3}$ is of type $\mathcal{D}_3$ using Remark 3.2.

By (d), $\mathbf{P}^k\sigma\mathbf{P}^k \in \mathbb{S}_m^\sigma$, i. e. $\mathbf{P}^k\sigma\mathbf{P}^k\sigma\mathbf{P}^{-k}\sigma^{-1}\mathbf{P}^{-k} = \sigma$, or $\sigma\mathbf{P}^k\sigma\mathbf{P}^{-k}\sigma^{-1} = \mathbf{P}^{-k}\sigma\mathbf{P}^k$. That is, $\sigma_1 \triangleright \sigma_2 = \sigma_3$. Analogously, $\sigma_1 \triangleright \sigma_3 = \sigma_2$ is proved using (e). To check that $\sigma_2 \triangleright \sigma_3 = \sigma_1$, note that $\sigma_2 \triangleright \sigma_3 = \mathbf{P}^k\sigma\mathbf{P}^{-k}\mathbf{P}^{-k}\sigma\mathbf{P}^k\mathbf{P}^k\sigma^{-1}\mathbf{P}^{-k} = \sigma$, because $\mathbf{P}^k\sigma\mathbf{P}^{-2k} = \mathbf{P}^k\sigma\mathbf{P}^k\mathbf{P}^{-3k} = \sigma B_1B_2 \in \mathbb{S}_m^\sigma$, by (a) and (d).

We now apply Corollary 3.8 and conclude that $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$. □

We shall need a few well-known results on symmetric groups.

Remark 3.11. (i) If ρ is a faithful representation of $\mathbb{S}_n$, then $\rho(\tau) \notin \mathbb{C}\,\mathrm{id}$, for every $\tau \in \mathbb{S}_n$, $\tau \neq \mathrm{id}$ (since $\mathbb{S}_n$ is centerless).

(ii) If $\rho = (\rho, W) \in \widehat{\mathbb{S}_n}$, with $\rho \neq \mathrm{sgn}$, then for any involution $\tau \in \mathbb{S}_n$ (i. e., $\tau^2 = \mathrm{id}$), there exists $w \in W - 0$ such that $\rho(\tau)w = w$ (otherwise $\rho(\tau) = -\,\mathrm{id}$).

Example 3.12. *Let $m \geq 6$. Let $\sigma \in \mathbb{S}_m$ of type $(1^{n_1}, 2^{n_2}, \dots, m^{n_m})$, $\mathcal{O}$ the conjugacy class of σ and $\rho \in \widehat{\mathbb{S}_m^\sigma}$. Assume that*

- $n_2 \geq 3$ *and*
- *there exists* j*, with* $j \geq 3$*, such that* $n_j \geq 1$.

Then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Proof. By Lemma 1.3, we may suppose that $q_{\sigma\sigma} = -1$. Assume that $(i_1\, i_2)$, $(i_3\, i_4)$ and $(i_5\, i_6)$ are three transpositions appearing in the decomposition of σ as a product of disjoint cycles. We define

$$x := (i_1\, i_2)(i_3\, i_4)(i_5\, i_6), \quad y := (i_1\, i_4)(i_3\, i_6)(i_2\, i_5), \quad z := (i_1\, i_6)(i_2\, i_3)(i_4\, i_5)$$

and $\alpha := x\sigma$. It is easy to see, using for instance Proposition 3.3, that

$$\sigma_1 := \sigma, \qquad \sigma_2 := y\alpha, \qquad \sigma_3 := z\alpha,$$

is of type $\mathcal{D}_3$. Then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$, by Corollary 3.8. Indeed, $\sigma^{-1} \in \mathcal{O}$, but $\sigma \neq \sigma^{-1}$ because σ has order > 2. $\square$

In the proof of the next Example, we need some notation for the induced representation. Let H be a subgroup of a finite group G of index k, $\phi_1, \dots, \phi_k$ the left cosets of H in G, with representatives $g_{\phi_1}, \dots, g_{\phi_k}$. Let $\theta = (\theta, W) \in \widehat{H}$, and $w_1, \dots w_r$ a basis of W. Set $V :=$span-$\{g_{\phi_i} w_j \,|\, 1 \leq i \leq k, 1 \leq j \leq r\}$. For i, j, with $1 \leq i \leq k$, $1 \leq j \leq r$ we define $\rho : G \to \mathrm{Aut}(V)$ by

$$\rho(g)(g_{\phi_i} w_j) = g_{\phi_l}\theta(h) w_j, \qquad \text{where } g g_{\phi_i} = g_{\phi_l} h, \text{ with } h \in H. \tag{3.23}$$

Thus $\rho = (\rho, V)$ is a representation of G and $\deg \rho = [G : H] \deg \theta$.

Example 3.13. *Let* $m \geq 12$*. Let* $\sigma \in \mathbb{S}_m$ *of type* $(1^{n_1}, 2^{n_2}, \dots, m^{n_m})$*,* $\mathcal{O}$ *the conjugacy class of* σ *and* $\rho \in \widehat{\mathbb{S}_m^\sigma}$*. If* $n_2 \geq 6$*, then* $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Proof. By Lemma 1.3, we may suppose that $q_{\sigma\sigma} = -1$. We denote the n_2 transpositions appearing in the decomposition of σ as product of disjoint cycles by $A_{1,2}, \dots, A_{n_2,2}$ and we define $A_2 = A_{1,2} \cdots A_{n_2,2}$. Let us suppose that $A_{1,2} = (i_1\, i_2)$, $A_{2,2} = (i_3\, i_4)$, $A_{3,2} = (i_5\, i_6)$, $A_{4,2} = (i_7\, i_8)$, $A_{5,2} = (i_9\, i_{10})$ and $A_{6,2} = (i_{11}\, i_{12})$. We define $x := (i_1\, i_2)(i_3\, i_4)(i_5\, i_6)(i_7\, i_8)(i_9\, i_{10})(i_{11}\, i_{12})$ and $\alpha := x\sigma$.

If there exists j, with $j \geq 3$, such that $n_j \geq 1$, then the result follows from Example 3.12. Assume that $n_j = 0$, for every $j \geq 3$, i. e. the type of σ is $(1^{n_1}, 2^{n_2})$. The centralizer of σ in $\mathbb{S}_m$ is $\mathbb{S}_m^\sigma = T_1 \times T_2$, with $T_1 \simeq \mathbb{S}_{n_1}$ and $T_2 = \Gamma \rtimes \Lambda$, with

$$\Gamma := \langle A_{1,2}, \dots, A_{n_2,2} \rangle, \quad \Lambda := \langle B_{1,2}, \dots, B_{n_2-1,2} \rangle.$$

Here $B_{l,2} := (i_{2l-1}\; i_{2l+1})(i_{2l}\; i_{2l+2})$, for $1 \leq l \leq n_2 - 1$. Note that $\Gamma \simeq (\mathbb{Z}/2)^{n_2}$ and $\Lambda \simeq \mathbb{S}_{n_2}$. Now, $\rho = \rho_1 \otimes \rho_2$, with $\rho_1 = (\rho_1, V_1) \in \widehat{T_1}$ and $\rho_2 = (\rho_2, V_2) \in \widehat{T_2}$.

For every t, $1 \le t \le n_2$, we define $\chi_t \in \widehat{\Gamma}$, by $\chi_t(A_{l,2}) = (-1)^{\delta_{t,l}}$, $1 \le l \le n_2$. Then, the irreducible representations of Γ are

$$\chi_{t_1,\dots,t_J} := \chi_{t_1} \dots \chi_{t_J}, \qquad 0 \le J \le n_2, \quad 1 \le t_1 < \dots < t_J \le n_2.$$

The case $J = 0$ corresponds to the trivial representation of Γ.

For every J, with $0 \le J \le n_2$, we denote $\chi_{(J)} := \chi_{1,\dots,J}$. The action of Λ on Γ induces a natural action of Λ on $\widehat{\Gamma}$, namely $(\lambda \cdot \chi)(A_{l,2}) := \chi(\lambda^{-1} A_{l,2} \lambda)$, $1 \le l \le n_2$, $\lambda \in \Lambda$. The orbit and the isotropy subgroup of $\chi_{(J)} \in \widehat{\Gamma}$ are

$$\mathcal{O}_{\chi_{(J)}} = \{\chi_{k_1,\dots,k_J} : 1 \le k_1 < \dots < k_J \le n_2\}, \tag{3.24}$$

$$\begin{aligned} \Lambda^{\chi_{(J)}} &= (\Lambda^{\chi_{(J)}})_1 \times (\Lambda^{\chi_{(J)}})_2 \\ &= \langle B_{1,2}, \dots, B_{J-1,2} \rangle \times \langle B_{J+1,2}, \dots, B_{n_2-1,2} \rangle \simeq \mathbb{S}_J \times \mathbb{S}_{n_2-J}. \end{aligned} \tag{3.25}$$

Thus, the characters $\chi_{(J)}$, $0 \le J \le n$, form a complete set of representatives of the orbits in $\widehat{\Gamma}$ under the action of Λ.

Since $\rho_2 \in \widehat{\Gamma \rtimes \Lambda}$, we have that $\rho_2 = \operatorname{Ind}_{\Gamma \rtimes \Lambda^{\chi_{(J)}}}^{\Gamma \rtimes \Lambda} \chi_{(J)} \otimes \mu$, with $\chi_{(J)}$ as above and $\mu = (\mu, W) \in \widehat{\Lambda^{\chi_{(J)}}}$ – see [S, Section 8.2]. By (3.25), $\mu = \mu_1 \otimes \mu_2$, with $\mu_l = (\mu_l, W_l) \in (\widehat{\Lambda^{\chi_{(J)}}})_l$, $l = 1, 2$. Let $\{\phi_1 = \Lambda^{\chi_{(J)}}, \dots, \phi_k\}$ the left cosets of $\Lambda^{\chi_{(J)}}$ in Λ, where $k = [\Lambda : \Lambda^{\chi_{(J)}}] = \frac{n_2!}{J!(n_2-J)!}$.

Note that

$$B_{1,2} = (i_1\ i_3)(i_2\ i_4), \quad B_{3,2} = (i_5\ i_7)(i_6\ i_8) \quad \text{and} \quad B_{5,2} = (i_9\ i_{11})(i_{10}\ i_{12}).$$

We define $B := B_{1,2}B_{3,2}B_{5,2}$. Notice that the order of B is 2.

Since $q_{\sigma\sigma} = -1$, then J is odd. We will consider two cases.

CASE (1): *assume that* $J \le 5$. Then, $B \notin \Lambda^{\chi_{(J)}}$. This implies that the left coset ϕ of $\Lambda^{\chi_{(J)}}$ in Λ containing B is not the trivial coset ϕ_1. We choose as representatives of the cosets ϕ_1 and ϕ to $g_{\phi_1} = \mathrm{id}$ and $g_\phi = B$, respectively. We define $v_2 := g_{\phi_1} w + g_\phi w$, with $w \in W - 0$. Notice that $Bg_{\phi_1} = g_\phi \,\mathrm{id}$ and $Bg_\phi = g_{\phi_1} \mathrm{id}$. Using (3.23), we have that

$$\begin{aligned} \rho_2(B)v_2 &= \rho_2(B)(g_{\phi_1} w) + \rho_2(B)(g_\phi w) \\ &= g_\phi \mu(\mathrm{id}) w + g_{\phi_1} \mu(\mathrm{id}) w = g_\phi w + g_{\phi_1} w = v_2. \end{aligned} \tag{3.26}$$

Let $v := v_1 \otimes v_2$, with $v_1 \in V_1 - 0$. Then,

(3.27)
$$\rho(B)v = (\rho_1 \otimes \rho_2)(\mathrm{id}, B)(v_1 \otimes v_2) = \rho_1(\mathrm{id})v_1 \otimes \rho_2(B)v_2 = v_1 \otimes v_2 = v,$$

by (3.26). We define $\sigma_1 := \sigma$,

$$\sigma_2 := (i_1\ i_6)(i_3\ i_8)(i_5\ i_{10})(i_7\ i_{12})(i_9\ i_2)(i_{11}\ i_4)\alpha,$$
$$\sigma_3 := (i_1\ i_{10})(i_3\ i_{12})(i_5\ i_2)(i_7\ i_4)(i_9\ i_6)(i_{11}\ i_8)\alpha,$$
$$\tau_1 := (i_1\ i_4)(i_3\ i_2)(i_5\ i_8)(i_7\ i_6)(i_9\ i_{12})(i_{11}\ i_{10})\alpha,$$
$$\tau_2 := (i_1\ i_8)(i_3\ i_6)(i_5\ i_{12})(i_7\ i_{10})(i_9\ i_4)(i_{11}\ i_2)\alpha,$$
$$\tau_3 := (i_1\ i_{12})(i_3\ i_{10})(i_5\ i_4)(i_7\ i_2)(i_9\ i_8)(i_{11}\ i_6)\alpha.$$

We can check by straightforward computations that (σ, τ) is of type $\mathcal{D}_3^{(2)}$. Let $g := (i_2\ i_4)(i_6\ i_8)(i_{10}\ i_{12})$; thus, $g \triangleright \sigma = \tau_1$. Moreover, $\tau_1 = \sigma B = g\sigma g$ and $\sigma_2\tau_2 = B = g\sigma_2\tau_2 g$. Then,

$$\rho(\tau_1)v = -v = \rho(g\sigma_1 g)v,$$

by (3.27). Therefore, $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$, by Theorem 3.7.

CASE (2): *assume that* $J \geq 7$. Then, $B \in \Lambda^{\chi(J)}$; moreover, $B \in (\Lambda^{\chi(J)})_1$. Also, $Bg_{\phi_1} = g_{\phi_1}B$.

Let $v_2 = g_{\phi_1}w$, with $w \in W - 0$. Since $W = W_1 \otimes W_2$, we may assume that $w = w_1 \otimes w_2$, with $w_1 \in W_1 - 0$ and $w_2 \in W_2 - 0$. Then, using (3.23),

$$\begin{aligned}\rho_2(B)v_2 &= \rho_2(B)(g_{\phi_1}w) = g_{\phi_1}\mu(B)w = g_{\phi_1}(\mu_1 \otimes \mu_2)(B, \mathrm{id})(w_1 \otimes w_2)\\ &= g_{\phi_1}\Big(\mu_1(B)(w_1) \otimes \mu_2(\mathrm{id})(w_2)\Big) = g_{\phi_1}\Big((\mu_1(B)(w_1) \otimes w_2\Big).\end{aligned}$$

Notice that $\mu_1 \in \widehat{(\Lambda^{\chi(J)})_1}$. Since $(\Lambda^{\chi(J)})_1 \simeq \mathbb{S}_J$, if $\mu_1 \neq \mathrm{sgn}$, with sgn the sign representation of $\mathbb{S}_J$, then there exists $w_1 \in W_1 - 0$ such that $\mu_1(B)(w_1) = w_1$, by Remark 3.11 (ii). In this case, we have

$$\rho_2(B)v_2 = g_{\phi_1}(\mu_1(B)(w_1) \otimes w_2) = g_{\phi_1}(w_1 \otimes w_2) = g_{\phi_1}w = v_2. \tag{3.28}$$

Taking $v := v_1 \otimes v_2$, with $v_1 \in V_1 - 0$, we have

$$\rho(B)v = (\rho_1 \otimes \rho_2)(\mathrm{id}, B)(v_1 \otimes v_2) = \rho_1(\mathrm{id})v_1 \otimes \rho_2(B)v_2 = v_1 \otimes v_2 = v,$$

by (3.28). Considering σ_i, τ_i, $1 \leq i \leq 3$, as in the previous case, the hypothesis of Corollary 3.8 hold. Therefore, $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

On the other hand, let us suppose that $\mu_1 = \mathrm{sgn}$. Let $w \in W$, with $w = w_1 \otimes w_2$, $w_1 \in W_1 - 0$ and $w_2 \in W_2 - 0$. Let $v_2 = g_{\phi_1}w$; since $\mu_1(B)(w_1) = -w_1$, we have $\rho_2(B)v_2 = -v_2$. Choosing $v := v_1 \otimes v_2$, with $v_1 \in V_1 - 0$, we have that

(3.29)
$$\rho(B)v = (\rho_1 \otimes \rho_2)(\mathrm{id}, B)(v_1 \otimes v_2) = \rho_1(\mathrm{id})v_1 \otimes \rho_2(B)v_2 = -v_1 \otimes v_2 = -v.$$

We define $\overline{\sigma_1} := \sigma$,

$$\begin{aligned}
\overline{\sigma_2} &:= (i_1\ i_6)(i_4\ i_7)(i_5\ i_{10})(i_8\ i_{11})(i_2\ i_9)(i_3\ i_{12})\alpha,\\
\overline{\sigma_3} &:= (i_1\ i_{10})(i_4\ i_{11})(i_2\ i_5)(i_3\ i_8)(i_6\ i_9)(i_7\ i_{12})\alpha,\\
\overline{\tau_1} &:= (i_1\ i_3)(i_2\ i_4)(i_5\ i_7)(i_6\ i_8)(i_9\ i_{11})(i_{10}\ i_{12})\alpha,\\
\overline{\tau_2} &:= (i_1\ i_7)(i_2\ i_{12})(i_3\ i_9)(i_4\ i_6)(i_5\ i_{11})(i_8\ i_{10})\alpha,\\
\overline{\tau_3} &:= (i_1\ i_{11})(i_2\ i_8)(i_3\ i_5)(i_4\ i_{10})(i_6\ i_{12})(i_7\ i_9)\alpha.
\end{aligned}$$

It can be shown that $(\overline{\sigma}, \overline{\tau})$ is of type $\mathcal{D}_3^{(2)}$. Let now $\overline{g} = (i_2\ i_3)(i_6\ i_7)(i_{10}\ i_{11})$; then, $\overline{g} \triangleright \sigma = \overline{\tau_1}$. Furthermore, $\overline{\tau_1} = B = \overline{g}\sigma\overline{g}$ and $\overline{\sigma_2}\,\overline{\tau_2} = \sigma B = \overline{g}\,\overline{\sigma_2}\,\overline{\tau_2}\,\overline{g}$. Then

$$\rho(\overline{\tau_1})v = -v = \rho(\overline{g}\sigma\overline{g})v \qquad \text{and} \qquad \rho(\overline{\sigma_2}\,\overline{\tau_2})v = v = \rho(\overline{g}\,\overline{\sigma_2}\,\overline{\tau_2}\,\overline{g})v,$$

by (3.29). Therefore, $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$, by Theorem 3.7. □

A way to obtain a family of type $\mathcal{D}_3$ is to start from a monomorphism $\rho : \mathbb{S}_3 \to G$ and to consider the image by ρ of the transpositions. Another way is as follows.

Remarks 3.14. Let G be a finite group and $z \in Z(G)$.

(i). Let $(\sigma_i)_{i\in\mathbb{Z}/3}$ be of type $\mathcal{D}_3$. Then $(z\sigma_i)_{i\in\mathbb{Z}/3}$ is also of type $\mathcal{D}_3$.

(ii). Let $(\sigma, \tau) = (\sigma_i)_{i\in\mathbb{Z}/3} \cup (\tau_i)_{i\in\mathbb{Z}/3}$ be a family of type $\mathcal{D}_3^{(2)}$. Then $(z\sigma, z\tau) = (z\sigma_i)_{i\in\mathbb{Z}/3} \cup (z\tau_i)_{i\in\mathbb{Z}/3}$ is also a family of type $\mathcal{D}_3^{(2)}$.

Here is a combination of these two ways.

Example 3.15. *Let p be a prime number and $q = p^m$, $m \in \mathbb{N}$, such that 3 divides $q-1$. Let $\omega \in \mathbb{F}_q$ be a primitive third root of 1.*

(i). If $c \in \mathbb{F}_q$, then $(\mu_i)_{i\in\mathbb{Z}/3}$, where $\mu_i = \begin{pmatrix} 0 & \omega^i \\ \omega^{2i}c & 0 \end{pmatrix}$, is a family of type $\mathcal{D}_3$ in $\mathbf{GL}(2, \mathbb{F}_q)$. If $c = -1$, then this is a family of type $\mathcal{D}_3$ in $\mathbf{SL}(2, \mathbb{F}_q)$. The orbit of μ_i is the set of matrices with minimal polynomial $T^2 - c$.

(ii). Let $N > 3$ be an integer and let $\mathbb{T}$ be the subgroup of diagonal matrices in $\mathbf{GL}(N, \mathbb{F}_q)$. Let $\lambda = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_N) \in \mathbb{T}$. Let $\mathcal{O}$ be the conjugacy class of λ. Assume that $\lambda_1 = -\lambda_2$ and let $c = \lambda_1^2$. Assume also that there exist i, j, with $3 \le i, j \le N$ such that $\lambda_i \neq \lambda_j$; say $i = 3$, $j = 4$, for simplicity of the exposition. Then $(\sigma_i)_{i\in\mathbb{Z}/3} \cup (\tau_i)_{i\in\mathbb{Z}/3}$, where

$$\sigma_i = \begin{pmatrix} \mu_i & 0 \\ 0 & \operatorname{diag}(\lambda_3, \lambda_4, \dots, \lambda_N) \end{pmatrix}, \qquad \tau_i = \begin{pmatrix} \mu_i & 0 \\ 0 & \operatorname{diag}(\lambda_4, \lambda_3, \dots, \lambda_N) \end{pmatrix},$$

is a family of type $\mathcal{D}_3^{(2)}$ in the orbit $\mathcal{O} \subset \mathbf{GL}(N, \mathbb{F}_q)$.

Let $\mathcal{W} = \mathbb{S}_N$ act on $\mathbb{T}$ in the natural way. Let $\chi : \mathbf{GL}(N, \mathbb{F}_q) \to \mathbb{C}^\times$ be a character; it restricts to an irreducible representation $(\chi, \mathbb{C})$ of the centralizer $\mathbf{GL}(N, \mathbb{F}_q)^{\sigma_0}$. Fix a group isomorphism $\varphi : \mathbb{F}_q^\times \to \mathbb{G}_{q-1} \subset \mathbb{C}^\times$, where $\mathbb{G}_{q-1}$ is the group of $(q-1)$-th roots of 1 in $\mathbb{C}$. Recall that $\chi = \varphi(\det^h)$ for some integer h. Thus the restriction of χ to $\mathbb{T}$ is $\mathcal{W}$-invariant.

Proposition 3.16. *Keep the notation above. Assume that $\chi(\lambda) = -1$. Then the dimension of the Nichols algebra $\mathfrak{B}(\mathcal{O}, \chi)$ is infinite.*

Proof. The result follows from Theorem 3.7. Indeed, hypothesis (H1) and (H2) clearly hold. The matrix $g = \begin{pmatrix} \mathrm{id}_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{N-4} \end{pmatrix}$ is an involution that satisfies $g \triangleright \sigma_0 = \tau_0$. Because of the explicit form of χ, $\chi(\sigma_0) = -1 = \chi(\tau_0)$, hence (H3) and (H4) hold. $\square$

This example can be adapted to the setting of semisimple orbits in finite groups of Lie type.

4. A technique from the symmetric group $\mathbb{S}_4$

The classification of the finite-dimensional Nichols algebras over $\mathbb{S}_4$, given in [AHS], relies on the fact (proved in *loc. cit.*) that some Nichols algebras $\mathfrak{B}(V_i \oplus V_j)$ have infinite dimension. According to the general strategy proposed in the present paper, each of these pairs (V_i, V_j) gives rise to a rack and a cocycle, and to a technique to discard Nichols algebras over other groups. Here we study one of these possibilities, and leave the others for a future publication.

The *octahedral rack* is the rack $X = \{1, 2, 3, 4, 5, 6\}$ given by the vertices of the octahedron with the operation of rack given by the "right-hand rule", i. e. if T_i is the orthogonal linear map that fixes i and rotates the orthogonal plane by an angle of $\pi/2$ with the right-hand rule (pointing the thumb to i), then we define $\triangleright : X \times X \to X$ by $i \triangleright j := T_i(j)$ – see Figure 1.

Explicitly,

$$\begin{array}{llllll}
1 \triangleright 1 = 1, & 2 \triangleright 1 = 3, & 3 \triangleright 1 = 4, & 4 \triangleright 1 = 5, & 5 \triangleright 1 = 2, & 6 \triangleright 1 = 1, \\
1 \triangleright 2 = 5, & 2 \triangleright 2 = 2, & 3 \triangleright 2 = 1, & 4 \triangleright 2 = 2, & 5 \triangleright 2 = 6, & 6 \triangleright 2 = 3, \\
1 \triangleright 3 = 2, & 2 \triangleright 3 = 6, & 3 \triangleright 3 = 3, & 4 \triangleright 3 = 1, & 5 \triangleright 3 = 3, & 6 \triangleright 3 = 4, \\
1 \triangleright 4 = 3, & 2 \triangleright 4 = 4, & 3 \triangleright 4 = 6, & 4 \triangleright 4 = 4, & 5 \triangleright 4 = 1, & 6 \triangleright 4 = 5, \\
1 \triangleright 5 = 4, & 2 \triangleright 5 = 1, & 3 \triangleright 5 = 5, & 4 \triangleright 5 = 6, & 5 \triangleright 5 = 5, & 6 \triangleright 5 = 2, \\
1 \triangleright 6 = 6, & 2 \triangleright 6 = 5, & 3 \triangleright 6 = 2, & 4 \triangleright 6 = 3, & 5 \triangleright 6 = 4, & 6 \triangleright 6 = 6.
\end{array}$$

Let G be a finite group, σ_1, σ_2, σ_3, σ_4, σ_5, $\sigma_6 \in G$ distinct elements and $\mathcal{O}$ the conjugacy class of σ_1 in G.

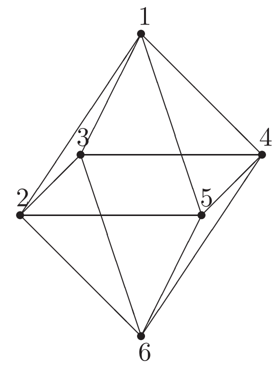

FIGURE 1. Octahedral rack.

Definition 4.1. We will say that $(\sigma_i)_{1\le i\le 6}$ is of *type* $\mathfrak{O}$ if the following holds

$$\sigma_i \triangleright \sigma_j = \sigma_{i\triangleright j}, \qquad 1 \le i, j \le 6.$$

Here and in the rest of this section, $\triangleright$ *in the subindex is the operation of rack in the octahedral rack.* In other words, $(\sigma_i)_{1\le i\le 6}$ is of type $\mathfrak{O}$ if and only if $\{\sigma_i \,|\, 1 \le i \le 6\}$ is isomorphic to the octahedral rack via $i \mapsto \sigma_i$.

Example 4.2. *Let* $m \ge 4$. *Let us consider in* $\mathbb{S}_m$ *the following 4-cycles*

$$\begin{aligned} &\widetilde{\sigma}_1 = (1\,2\,3\,4), \qquad \widetilde{\sigma}_2 = (1\,2\,4\,3), \qquad \widetilde{\sigma}_3 = (1\,3\,2\,4),\\ &\widetilde{\sigma}_4 = (1\,3\,4\,2), \qquad \widetilde{\sigma}_5 = (1\,4\,2\,3), \qquad \widetilde{\sigma}_6 = (1\,4\,3\,2). \end{aligned} \tag{4.1}$$

It is easy to see that $(\widetilde{\sigma}_i)_{1\le i\le 6}$ *satisfy the relations given in the previous definition. Thus,* $(\widetilde{\sigma}_i)_{1\le i\le 6}$ *is of type* $\mathfrak{O}$.

Let $\chi_- \in \widehat{\mathbb{S}_4^{\widetilde{\sigma}_1}}$ be given by $\chi_-(1\,2\,3\,4) = -1$. The goal of this Section is to apply the next result, cf. [AHS, Theor. 4.7].

Theorem 4.3. *The Nichols algebra* $\mathfrak{B}\left(M(\mathcal{O}_4^4, \chi_-) \oplus M(\mathcal{O}_4^4, \chi_-)\right)$ *has infinite dimension.* $\square$

Remark 4.4. We note that $M(\mathcal{O}_4^4, \chi_-) \oplus M(\mathcal{O}_4^4, \chi_-) \simeq (\mathbb{C}Y, \mathfrak{q})$ as braided vector spaces, where

- $Y = \{x_i, y_j \mid 1 \le i, j \le 6\} \simeq X^{(2)}$, see Definition 1.1;
- $\mathfrak{q}$ is the constant cocycle $\mathfrak{q} \equiv -1$.

Proof. We define

$$\begin{aligned} &\widetilde{\sigma}_1 := (1\,2\,3\,4) =: \widetilde{\tau}_1, \qquad \widetilde{\sigma}_2 := (1\,2\,4\,3) =: \widetilde{\tau}_2, \qquad \widetilde{\sigma}_3 := (1\,3\,2\,4) =: \widetilde{\tau}_3,\\ &\widetilde{\sigma}_4 := (1\,3\,4\,2) =: \widetilde{\tau}_4, \qquad \widetilde{\sigma}_5 := (1\,4\,2\,3) =: \widetilde{\tau}_5, \qquad \widetilde{\sigma}_6 := (1\,4\,3\,2) =: \widetilde{\tau}_6. \end{aligned}$$

We will denote by $(\widetilde{\sigma}_j)_{1\leq j\leq 6}$ (resp. $(\widetilde{\tau}_j)_{1\leq j\leq 6}$) the first copy (resp. the second copy) of $\mathcal{O}_4^4$, with system of left cosets representatives of $\mathbb{S}_4^{(1\,2\,3\,4)}$ given by $\widetilde{g}_1=\widetilde{g}_7=\widetilde{\sigma}_1$, $\widetilde{g}_2=\widetilde{g}_8=\widetilde{\sigma}_5$, $\widetilde{g}_3=\widetilde{g}_9=\widetilde{\sigma}_2$, $\widetilde{g}_4=\widetilde{g}_{10}=\widetilde{\sigma}_3$, $\widetilde{g}_5=\widetilde{g}_{11}=\widetilde{\sigma}_4$, $\widetilde{g}_6=\widetilde{g}_{12}=\widetilde{\sigma}_2^2\widetilde{\sigma}_1$. The map $M(\mathcal{O}_4^4,\chi_-)\oplus M(\mathcal{O}_4^4,\chi_-)\to(\mathbb{C}Y,\mathfrak{q})$ given by

$$\widetilde{g}_i\mapsto x_i \quad\text{and}\quad \widetilde{g}_{i+6}\mapsto y_i,\qquad 1\leq i\leq 6,$$

is an isomorphism of braided vector spaces. □

Proposition 4.5. *A family $(\sigma_i)_{1\leq i\leq 6}$ of distinct elements in G is of type $\mathfrak{O}$ if and only if the following identities hold:*

(4.2)
$$\sigma_1\triangleright\sigma_2=\sigma_5,\quad \sigma_1\triangleright\sigma_3=\sigma_2,\quad \sigma_1\triangleright\sigma_4=\sigma_3,\quad \sigma_1\triangleright\sigma_5=\sigma_4,\quad \sigma_1\triangleright\sigma_6=\sigma_6,$$

(4.3)
$$\sigma_2\triangleright\sigma_1=\sigma_3,\quad \sigma_2\triangleright\sigma_3=\sigma_6,\quad \sigma_2\triangleright\sigma_4=\sigma_4,\quad \sigma_2\triangleright\sigma_5=\sigma_1,\quad \sigma_2\triangleright\sigma_6=\sigma_5.$$

Proof. If we apply $\sigma_1\triangleright\_\_$ to the relations in (4.3), then we obtain the relations $\sigma_5\triangleright\sigma_j=\sigma_{5\triangleright j}$, $1\leq j\leq 6$, because $\sigma_1\triangleright\sigma_2=\sigma_5$. Analogously, we obtain the relations $\sigma_i\triangleright\sigma_j=\sigma_{i\triangleright j}$, $1\leq j\leq 6$, for $i=3$, 4; and the relations $\sigma_6\triangleright\sigma_j=\sigma_{6\triangleright j}$, $1\leq j\leq 6$, follow by applying $\sigma_5\triangleright\_\_$ to the ones in (4.3). □

Lemma 4.6. *If $(\sigma_i)_{1\leq i\leq 6}$ is of type $\mathfrak{O}$, then*

(i) $\sigma_1^4=\sigma_2^4=\sigma_3^4=\sigma_4^4=\sigma_5^4=\sigma_6^4$,
(ii) $\sigma_1\sigma_6=\sigma_2\sigma_4=\sigma_3\sigma_5$,
(iii) $\sigma_2^2\sigma_5^2=\sigma_1^3\sigma_6=\sigma_3^2\sigma_2^2$,
(iv) $\sigma_5^2\sigma_2^2=\sigma_1\sigma_6^3=\sigma_2^2\sigma_3^2$.

Proof. (i). Since $\sigma_i\triangleright(\sigma_i\triangleright(\sigma_i(\triangleright(\sigma_i\triangleright\sigma_j))))=\sigma_j$, then $\sigma_i^4\in G^{\sigma_j}$, $1\leq i,j\leq 6$. Hence $\sigma_1^4=(\sigma_3\sigma_2\sigma_3^{-1})^4=\sigma_3\sigma_2^4\sigma_3^{-1}=\sigma_2^4$, and the rest is similar. (ii). By Definition 4.1, we see that

$$\sigma_3\sigma_5=\sigma_3\sigma_1\sigma_2\sigma_1^{-1}=\sigma_3\sigma_2\sigma_5\sigma_2^{-1}\sigma_2\sigma_1^{-1}=\sigma_2\sigma_1\sigma_5\sigma_1^{-1}=\sigma_2\sigma_4,$$
$$\sigma_3\sigma_5=\sigma_3\sigma_2\sigma_6\sigma_2^{-1}=\sigma_3\sigma_6\sigma_5\sigma_6^{-1}\sigma_6\sigma_2^{-1}=\sigma_6\sigma_2\sigma_5\sigma_2^{-1}=\sigma_6\sigma_1.$$

Then, $\sigma_1\sigma_6=\sigma_2\sigma_4=\sigma_3\sigma_5$, as claimed.

(iii). By (ii), we have that

$$\sigma_2^2\sigma_5^2=\sigma_2\sigma_5\sigma_1\sigma_5=\sigma_5\sigma_1\sigma_1\sigma_5=\sigma_5\sigma_1\sigma_4\sigma_1=\sigma_5\sigma_3\sigma_1^2=\sigma_1\sigma_6\sigma_1^2=\sigma_1^3\sigma_6.$$

Then, $\sigma_2^2\sigma_5^2=\sigma_1^3\sigma_6$. We apply $\sigma_1\triangleright(\sigma_1\triangleright(\sigma_1\triangleright\_\_))$ to the last expression and we have $\sigma_3^2\sigma_2^2=\sigma_1^3\sigma_6$.

(iv) follows from (iii) applying $\sigma_2\triangleright(\sigma_2\triangleright\_\_)$. □

Definition 4.7. Let σ_i, $\tau_i \in G$, $1 \leq i \leq 6$, all distinct. We say that (σ, τ) is *of type* $\mathfrak{D}^{(2)}$ if $(\sigma_i)_{1\leq i\leq 6}$ and $(\tau_j)_{1\leq j\leq 6}$ are both of type $\mathfrak{D}$, and

$$\sigma_i \triangleright \tau_j = \tau_{i\triangleright j}, \qquad \tau_i \triangleright \sigma_j = \sigma_{i\triangleright j}, \qquad 1 \leq i, j \leq 6. \tag{4.4}$$

Lemma 4.8. *If (σ, τ) is of type $\mathfrak{D}^{(2)}$, then*

(i) $\sigma_1\tau_6 = \sigma_6\tau_1 = \sigma_2\tau_4 = \sigma_4\tau_2 = \sigma_3\tau_5 = \sigma_5\tau_3$,
(ii) $\sigma_j^{-1}\tau_j = \sigma_1^{-1}\tau_1$, $2 \leq j \leq 6$,
(iii) $\tau_2^{-2}\sigma_5\tau_5 = \tau_1^{-1}\sigma_6$,
(iv) $\tau_2^{-2}\sigma_3\tau_3 = \sigma_1\tau_6^{-1}$,
(v) $\sigma_2^{-2}\sigma_5\tau_5 = \sigma_1^{-2}\tau_1\sigma_6$,
(vi) $\sigma_2^{-2}\sigma_3\tau_3 = \tau_1\sigma_6^{-1}$.

Proof. (i). First,

$$\sigma_1\tau_6 = \sigma_1\, \sigma_2\tau_3\sigma_2^{-1} = \tau_3\sigma_2\tau_3^{-1}\, \tau_3\sigma_6\tau_3^{-1}\, \tau_3\sigma_2^{-1} = \tau_3\, \sigma_2\sigma_6\sigma_2^{-1} = \tau_3\sigma_5 = \sigma_5\tau_3. \tag{4.5}$$

Applying now $\sigma_2 \triangleright \underline{\ \ }$ to (4.5) we get $\sigma_3\tau_5 = \tau_6\sigma_1$. Applying $\sigma_2 \triangleright \underline{\ \ }$ to this last identity, we have $\sigma_6\tau_1 = \tau_5\sigma_3$. The rest is similar.

(ii). By (i) and Lemma 4.6 (ii) for $(\tau_i)_{1\leq i\leq 6}$, we have that

$$\sigma_2^{-1}\tau_2 = \sigma_2^{-1}\tau_4^{-1}\tau_4\tau_2 = \sigma_1^{-1}\tau_6^{-1}\tau_1\tau_6 = \sigma_1^{-1}\tau_1.$$

The other relations can be obtained in an analogous way.

(iii). It is easy to see that

$$\begin{aligned}\tau_2^{-2}\sigma_5\tau_5 &= \tau_2^{-4}\tau_2\tau_2\tau_5\sigma_5 = \tau_1^{-4}\tau_2\tau_5\tau_1\sigma_5 = \tau_1^{-4}\tau_5\tau_1\tau_1\sigma_5\\ &= \tau_1^{-4}\tau_5\tau_1\sigma_4\tau_1 = \tau_1^{-4}\tau_5\sigma_3\tau_1\tau_1 = \tau_1^{-4}\tau_1\sigma_6\tau_1^2 = \tau_1^{-1}\sigma_6.\end{aligned}$$

(iv) follows from (iii) applying $\sigma_2 \triangleright (\sigma_2 \triangleright \underline{\ \ })$.

(v). Clearly,

$$\begin{aligned}\sigma_2^{-2}\sigma_5\tau_5 &= \sigma_2^{-4}\sigma_2\sigma_2\sigma_5\tau_5 = \sigma_1^{-4}\sigma_2\sigma_5\sigma_1\tau_5 = \sigma_1^{-4}\sigma_5\sigma_1\sigma_1\tau_5 = \sigma_1^{-4}\sigma_5\sigma_1\tau_4\sigma_1\\ &= \sigma_1^{-4}\sigma_5\tau_3\sigma_1\sigma_1 = \sigma_1^{-4}\sigma_1\tau_6\sigma_1\sigma_1 = \sigma_1^{-1}\tau_6 = \sigma_1^{-2}\tau_1\sigma_6.\end{aligned}$$

(vi) follows from (v) applying $\sigma_2 \triangleright (\sigma_2 \triangleright \underline{\ \ })$. $\square$

4.1. **Applications.** Let G be a finite group, $\mathcal{O}$ a conjugacy class of G. Let $(\sigma_i)_{1\leq i\leq 6} \subset \mathcal{O}$ be of type $\mathfrak{D}$. We define

$$g_1 := \sigma_1, \quad g_2 := \sigma_5, \quad g_3 := \sigma_2, \quad g_4 := \sigma_3, \quad g_5 := \sigma_4, \quad g_6 := \sigma_2^2\sigma_1; \tag{4.6}$$

then, $\sigma_i = g_i \triangleright \sigma_1$, $1 \le i \le 6$. It is easy to see that following relations hold

$$\begin{array}{lll}
\sigma_1 g_1 = g_1\sigma_1, & \sigma_1 g_2 = g_5\sigma_1, & \sigma_1 g_3 = g_2\sigma_1, \\
\sigma_2 g_1 = g_3\sigma_1, & \sigma_2 g_2 = g_2\sigma_1, & \sigma_2 g_3 = g_6\sigma_1^{-1}, \\
\sigma_3 g_1 = g_4\sigma_1, & \sigma_3 g_2 = g_1\sigma_6, & \sigma_3 g_3 = g_3\sigma_1, \\
\sigma_4 g_1 = g_5\sigma_1, & \sigma_4 g_2 = g_2\sigma_6, & \sigma_4 g_3 = g_1\sigma_6, \\
\sigma_5 g_1 = g_2\sigma_1, & \sigma_5 g_2 = g_6\sigma_1^{-2}\sigma_6, & \sigma_5 g_3 = g_3\sigma_6, \\
\sigma_6 g_1 = g_1\sigma_6, & \sigma_6 g_2 = g_3\sigma_6, & \sigma_6 g_3 = g_4\sigma_6,
\end{array}$$

$$\begin{array}{lll}
\sigma_1 g_4 = g_3\sigma_1, & \sigma_1 g_5 = g_4\sigma_1, & \sigma_1 g_6 = g_6\sigma_6, \\
\sigma_2 g_4 = g_4\sigma_6, & \sigma_2 g_5 = g_1\sigma_6, & \sigma_2 g_6 = g_5\sigma_6^3, \\
\sigma_3 g_4 = g_6\sigma_6^{-1}, & \sigma_3 g_5 = g_5\sigma_6, & \sigma_3 g_6 = g_2\sigma_1^3, \\
\sigma_4 g_4 = g_4\sigma_1, & \sigma_4 g_5 = g_6\sigma_1\sigma_6^{-2}, & \sigma_4 g_6 = g_3\sigma_1^2\sigma_6, \\
\sigma_5 g_4 = g_1\sigma_6, & \sigma_5 g_5 = g_5\sigma_1, & \sigma_5 g_6 = g_4\sigma_1\sigma_6^2, \\
\sigma_6 g_4 = g_5\sigma_6, & \sigma_6 g_5 = g_2\sigma_6, & \sigma_6 g_6 = g_6\sigma_1.
\end{array}$$

Let $\rho = (\rho, V) \in \widehat{G^{\sigma_1}}$ and $v \in V - 0$. Assume that v is an eigenvector of $\rho(\sigma_6)$ with eigenvalue λ. We define $W :=$ span- $\{g_i v \,|\, 1 \le i \le 6\}$. Then, W is a braided vector subspace of $M(\mathcal{O}, \rho)$.

Lemma 4.9. *Let $(\sigma_i)_{1\le i\le 6}$, $(g_i)_{1\le i\le 6}$, $(\rho, V) \in \widehat{G^{\sigma_1}}$, W, λ as above. Assume that $q_{\sigma_1\sigma_1} = \lambda = -1$. Then $W \simeq M(\mathcal{O}_4^4, \chi_-)$ as braided vector spaces.*

Proof. Since $q_{\sigma_1\sigma_1} = -1$ we have that $\rho(\sigma_i^4) = \mathrm{id}$, $1 \le i \le 6$, from Lemma (4.6) (i). Let $\widetilde{\sigma}_i$ be as in (4.1). If we choose

$$\widetilde{g}_1 = \widetilde{\sigma}_1, \quad \widetilde{g}_2 = \widetilde{\sigma}_5, \quad \widetilde{g}_3 = \widetilde{\sigma}_2, \quad \widetilde{g}_4 = \widetilde{\sigma}_3, \quad \widetilde{g}_5 = \widetilde{\sigma}_4, \quad \widetilde{g}_6 = \widetilde{\sigma}_2^2\widetilde{\sigma}_1,$$

then $\widetilde{g}_i \triangleright \widetilde{\sigma}_1 = \widetilde{\sigma}_i$, $1 \le i \le 6$. Thus, $M(\mathcal{O}_4^4, \chi_-) =$ span-$\{\widetilde{g}_i v_0, \,|\, 1 \le i \le 6\}$, with $v_0 \in V_0 - 0$, where V_0 is the vector space affording the representation χ_- of $\mathbb{S}_4^{(1\,2\,3\,4)}$. Now, the map $W \to M(\mathcal{O}_4^4, \chi_-)$ given by $g_i v \mapsto \widetilde{g}_i v_0$, $1 \le i \le 6$, is an isomorphism of braided vector spaces. $\square$

The next lemma is needed for the main result of the section.

Lemma 4.10. *Let σ_i, τ_i, $1 \le i \le 6$, be distinct elements in G, $\mathcal{O}$ a conjugacy class of G. Assume that $(\sigma, \tau) \subseteq \mathcal{O}$ is of type $\mathfrak{O}^{(2)}$, with $g \in G$ such that $g \triangleright \sigma_1 = \tau_1$. Let*

$$\begin{array}{llll}
g_1 := \sigma_1, & g_2 := \sigma_5, & g_3 := \sigma_2, & g_4 := \sigma_3, \\
g_5 := \sigma_4, & g_6 := \sigma_2^2\sigma_1, & g_7 := g\sigma_1, & g_8 := \tau_5 g, \\
g_9 := \tau_2 g, & g_{10} := \tau_3 g, & g_{11} := \tau_4 g, & g_{12} := \tau_2^2 g\sigma_1.
\end{array} \tag{4.7}$$

Then, the following relations hold:

$$\begin{array}{lll}
\tau_1 g_7 = g_7\sigma_1, & \tau_1 g_8 = g_{11}\sigma_1, & \tau_1 g_9 = g_8\sigma_1,\\
\tau_2 g_7 = g_9\sigma_1, & \tau_2 g_8 = g_8\sigma_1, & \tau_2 g_9 = g_{12}\sigma_1^{-1},\\
\tau_3 g_7 = g_{10}\sigma_1, & \tau_3 g_8 = g_7 g^{-1}\tau_6 g, & \tau_3 g_9 = g_9\sigma_1,\\
\tau_4 g_7 = g_{11}\sigma_1, & \tau_4 g_8 = g_8 g^{-1}\tau_6 g, & \tau_4 g_9 = g_7 g^{-1}\tau_6 g,\\
\tau_5 g_7 = g_8\sigma_1, & \tau_5 g_8 = g_{12}\sigma_1^{-2} g^{-1}\tau_6 g, & \tau_5 g_9 = g_9 g^{-1}\tau_6 g,\\
\tau_6 g_7 = g_7 g^{-1}\tau_6 g, & \tau_6 g_8 = g_9 g^{-1}\tau_6 g, & \tau_6 g_9 = g_{10} g^{-1}\tau_6 g,
\end{array}$$

$$\begin{array}{lll}
\tau_1 g_{10} = g_9\sigma_1, & \tau_1 g_{11} = g_{10}\sigma_1, & \tau_1 g_{12} = g_{12} g^{-1}\tau_6 g,\\
\tau_2 g_{10} = g_{10} g^{-1}\tau_6 g, & \tau_2 g_{11} = g_7 g^{-1}\tau_6 g, & \tau_2 g_{12} = g_{11}(g^{-1}\tau_6 g)^3,\\
\tau_3 g_{10} = g_{12}(g^{-1}\tau_6 g)^{-1}, & \tau_3 g_{11} = g_{11} g^{-1}\tau_6 g, & \tau_3 g_{12} = g_8\sigma_1^3,\\
\tau_4 g_{10} = g_{10}\sigma_1, & \tau_4 g_{11} = g_{12}\sigma_1(g^{-1}\tau_6 g)^{-2}, & \tau_4 g_{12} = g_9\sigma_1^2 g^{-1}\tau_6 g,\\
\tau_5 g_{10} = g_7 g^{-1}\tau_6 g, & \tau_5 g_{11} = g_{11}\sigma_1, & \tau_5 g_{12} = g_{10}\sigma_1(g^{-1}\tau_6 g)^2,\\
\tau_6 g_{10} = g_{11} g^{-1}\tau_6 g, & \tau_6 g_{11} = g_8 g^{-1}\tau_6 g, & \tau_6 g_{12} = g_{12}\sigma_1,
\end{array}$$

$$\begin{array}{lll}
\sigma_1 g_7 = g_7 g^{-1}\sigma_1 g, & \sigma_1 g_8 = g_{11} g^{-1}\sigma_1 g, & \sigma_1 g_9 = g_8 g^{-1}\sigma_1 g,\\
\sigma_2 g_7 = g_9 g^{-1}\sigma_1 g, & \sigma_2 g_8 = g_8 g^{-1}\sigma_1 g, & \sigma_2 g_9 = g_{12}\sigma_1^{-2}(g^{-1}\sigma_1 g),\\
\sigma_3 g_7 = g_{10} g^{-1}\sigma_1 g, & \sigma_3 g_8 = g_7 g^{-1}\sigma_6 g, & \sigma_3 g_9 = g_9 g^{-1}\sigma_1 g,\\
\sigma_4 g_7 = g_{11} g^{-1}\sigma_1 g, & \sigma_4 g_8 = g_8 g^{-1}\sigma_6 g, & \sigma_4 g_9 = g_7 g^{-1}\sigma_6 g,\\
\sigma_5 g_7 = g_8 g^{-1}\sigma_1 g, & \sigma_5 g_8 = g_{12}\sigma_1^{-2} g^{-1}\sigma_6 g, & \sigma_5 g_9 = g_9 g^{-1}\sigma_6 g,\\
\sigma_6 g_7 = g_7 g^{-1}\sigma_6 g, & \sigma_6 g_8 = g_9 g^{-1}\sigma_6 g, & \sigma_6 g_9 = g_{10} g^{-1}\sigma_6 g,
\end{array}$$

$$\begin{array}{lll}
\sigma_1 g_{10} = g_9 g^{-1}\sigma_1 g, & \sigma_1 g_{11} = g_{10} g^{-1}\sigma_1 g, & \sigma_1 g_{12} = g_{12} g^{-1}\sigma_6 g,\\
\sigma_2 g_{10} = g_{10} g^{-1}\sigma_6 g, & \sigma_2 g_{11} = g_7 g^{-1}\sigma_6 g, & \sigma_2 g_{12} = g_{11}\gamma_{2,12},\\
\sigma_3 g_{10} = g_{12}\gamma_{3,10}, & \sigma_3 g_{11} = g_{11} g^{-1}\sigma_6 g, & \sigma_3 g_{12} = g_8\sigma_1^2(g^{-1}\sigma_1 g),\\
\sigma_4 g_{10} = g_{10} g^{-1}\sigma_1 g, & \sigma_4 g_{11} = g_{12}\gamma_{4,11}, & \sigma_4 g_{12} = g_9\sigma_1^2 g^{-1}\sigma_6 g,\\
\sigma_5 g_{10} = g_7 g^{-1}\sigma_6 g, & \sigma_5 g_{11} = g_{11} g^{-1}\sigma_1 g, & \sigma_5 g_{12} = g_{10}\gamma_{5,12},\\
\sigma_6 g_{10} = g_{11} g^{-1}\sigma_6 g, & \sigma_6 g_{11} = g_8 g^{-1}\sigma_6 g, & \sigma_6 g_{12} = g_{12} g^{-1}\sigma_1 g,
\end{array}$$

where $\gamma_{2,12} = \sigma_1^2(g^{-1}\sigma_1 g)^{-2}(g^{-1}\sigma_6 g)^3$, $\gamma_{3,10} = \sigma_1^{-2}(g^{-1}\sigma_1 g)^2(g^{-1}\sigma_6 g)^{-1}$, $\gamma_{4,11} = \sigma_1^{-2}(g^{-1}\sigma_1 g)^3(g^{-1}\sigma_6 g)^{-2}$ *and* $\gamma_{5,12} = \sigma_1^2(g^{-1}\sigma_1 g)^{-1}(g^{-1}\sigma_6 g)^2$,

$$\begin{array}{lll}
\tau_1 g_1 = g_1\tau_1, & \tau_1 g_2 = g_5\tau_1, & \tau_1 g_3 = g_2\tau_1,\\
\tau_2 g_1 = g_3\tau_1, & \tau_2 g_2 = g_2\tau_1, & \tau_2 g_3 = g_6\sigma_1^{-2}\tau_1,\\
\tau_3 g_1 = g_4\tau_1, & \tau_3 g_2 = g_1\tau_6, & \tau_3 g_3 = g_3\tau_1,\\
\tau_4 g_1 = g_5\tau_1, & \tau_4 g_2 = g_2\tau_6, & \tau_4 g_3 = g_1\tau_6,\\
\tau_5 g_1 = g_2\tau_1, & \tau_5 g_2 = g_6\sigma_1^{-2}\tau_6, & \tau_5 g_3 = g_3\tau_6,\\
\tau_6 g_1 = g_1\tau_6, & \tau_6 g_2 = g_3\tau_6, & \tau_6 g_3 = g_4\tau_6,
\end{array}$$

$$\begin{array}{lll}
\tau_1 g_4 = g_3\tau_1, & \tau_1 g_5 = g_4\tau_1, & \tau_1 g_6 = g_6\tau_6,\\
\tau_2 g_4 = g_4\tau_6, & \tau_2 g_5 = g_1\tau_6, & \tau_2 g_6 = g_5\sigma_1^3\tau_1\sigma_6,\\
\tau_3 g_4 = g_6\sigma_1^{-1}\tau_1\sigma_6^{-1}, & \tau_3 g_5 = g_5\tau_6, & \tau_3 g_6 = g_2\sigma_1^2\tau_1,\\
\tau_4 g_4 = g_4\tau_1, & \tau_4 g_5 = g_6\tau_1\sigma_6^{-2}, & \tau_4 g_6 = g_3\sigma_1\tau_1\sigma_6,\\
\tau_5 g_4 = g_1\tau_6, & \tau_5 g_5 = g_5\tau_1, & \tau_5 g_6 = g_4\tau_1\sigma_6^2,\\
\tau_6 g_4 = g_5\tau_6, & \tau_6 g_5 = g_2\tau_6, & \tau_6 g_6 = g_6\tau_1.
\end{array}$$

Proof. The proof follows by straightforward computations, Lemma 4.6 for σ and τ, and Lemma 4.8. □

Here is the main result of this section.

Theorem 4.11. *Let σ_i, $\tau_i \in G$, $1 \le i \le 6$, distinct elements in G, $\mathcal{O}$ a conjugacy class of G and $\rho = (\rho, V) \in \widehat{G^{\sigma_1}}$. Let us suppose that*

(H1) *(σ, τ) is of type $\mathfrak{D}^{(2)}$,*
(H2) *$(\sigma, \tau) \subseteq \mathcal{O}$, with $g \in G$ such that $g \triangleright \sigma_1 = \tau_1$,*
(H3) *$q_{\sigma_1\sigma_1} = -1$,*

there exists $v \in V - 0$ such that

(H4) *$\rho(\sigma_6)v = -v$,*
(H5) *$\rho(\tau_1)v = -v$,*

and there exists $w \in V - 0$ such that

(H6) *$\rho(g^{-1}\sigma_1 g)w = -w$,*
(H7) *$\rho(g^{-1}\sigma_6 g)w = -w$,*

Then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Proof. Let $g_j \in G$, $1 \le j \le 12$, as in (4.7). Then, $g_j \triangleright \sigma_1 = \sigma_j$, $1 \le j \le 6$, and $g_j \triangleright \sigma_1 = \tau_{j-6}$, $7 \le j \le 12$. By Lemma 4.10, we have that

(a) if $1 \le i, j \le 6$, then $g_{i\triangleright j}^{-1}\sigma_i g_j = \sigma_1^r\sigma_6^s$, with $r + s$ odd,
(b) if $7 \le i, j \le 12$, then $g_{i\triangleright j}^{-1}\tau_{i-6} g_j = \sigma_1^r(g^{-1}\tau_6 g)^s$, with $r + s$ odd,

(c) if $1 \leq i \leq 6$ and $7 \leq j \leq 12$, then $g_{i \triangleright j}^{-1} \sigma_i g_j = \sigma_1^r (g^{-1}\sigma_1 g)^s (g^{-1}\sigma_6 g)^t$, with $r+s+t$ odd,

(d) if $1 \leq j \leq 6$ and $7 \leq i \leq 12$, then $g_{i \triangleright j}^{-1} \tau_{i-6} g_j = \sigma_1^r \tau_1^s \sigma_6^t$, with $r+s+t$ odd, because $\tau_6 = \sigma_1^{-1}\tau_1\sigma_6$.

Let $W :=$ span-$\{g_i v, \,|\, 1 \leq i \leq 6\}$ and $W' :=$ span-$\{g_i w, \,|\, 7 \leq i \leq 12\}$, with $v, w \in V - 0$, where v satisfies (H4)-(H5) and w satisfies (H6)-(H7). Then, W and W' are braided vector subspaces of $M(\mathcal{O}, \rho)$. We will prove that

$$W \oplus W' \simeq M(\mathcal{O}_4^4, \chi_-) \oplus M(\mathcal{O}_4^4, \chi_-),$$

as braided vector spaces. Hence $\dim \mathfrak{B}(W \oplus W') = \infty$, by Theorem 4.3, and the result follows from Lemma 1.2.

By Remark 4.4, we only need to see that the isomorphism of linear vector spaces $W \oplus W' \to M(\mathcal{O}_4^4, \chi_-) \oplus M(\mathcal{O}_4^4, \chi_-)$ given by

$$g_i v \mapsto \widetilde{g}_i \quad \text{and} \quad g_{i+6} w \mapsto \widetilde{g}_{i+6} \qquad 1 \leq i \leq 6,$$

respects the braiding, and this is just a matter of the cocycle. For this, we compute explicitly the braiding in the basis $\{g_i v, \, g_{j+6} w, \,|\, 1 \leq i, j \leq 6\}$ of $W \oplus W'$.

By (a), (H3) and (H4), if $1 \leq i, j \leq 6$, then

$$c(g_i v \otimes g_j v) = g_{i \triangleright j} \rho(g_{i \triangleright j}^{-1} \sigma_i g_j)(v) \otimes g_i v = -g_{i \triangleright j} v \otimes g_i v.$$

From Lemma 4.8 (i), $\tau_6 = \sigma_1^{-1}\tau_1\sigma_6$. Thus, $g^{-1}\tau_6 g = (g^{-1}\sigma_1 g)^{-1}\sigma_1(g^{-1}\sigma_6 g)$. By (b), (H3), (H6) and (H7), if $7 \leq i, j \leq 12$, then

$$c(g_i w \otimes g_j w) = g_{i \triangleright j} \rho(g_{i \triangleright j}^{-1} \tau_{i-6} g_j)(w) \otimes g_i w = -g_{i \triangleright j} w \otimes g_i w.$$

By (c), (H3), (H6) and (H7), if $1 \leq i \leq 6$ and $7 \leq j \leq 12$, then

$$c(g_i v \otimes g_j w) = g_{i \triangleright j} \rho(g_{i \triangleright j}^{-1} \sigma_i g_j)(w) \otimes g_i v = -g_{i \triangleright j} w \otimes g_i v.$$

By (d), (H3), (H4) and (H5), if $1 \leq j \leq 6$ and $7 \leq i \leq 12$, then

$$c(g_i w \otimes g_j v) = g_{i \triangleright j} \rho(g_{i \triangleright j}^{-1} \tau_{i-6} g_j)(v) \otimes g_i w = -g_{i \triangleright j} v \otimes g_i w.$$

This completes the proof. $\square$

As an immediate consequence we have the following result.

Corollary 4.12. *Let σ_i, $\tau_i \in G$, $1 \leq i \leq 6$ all distinct, $\mathcal{O}$ a conjugacy class of G and $\rho = (\rho, V) \in \widehat{G^{\sigma_1}}$ with $q_{\sigma_1\sigma_1} = -1$. Assume that $(\sigma, \tau) \subseteq \mathcal{O}$ is of type $\mathfrak{D}^{(2)}$. If $\sigma_6 = \sigma_1^d$ and $\tau_1 = \sigma_1^e$ for some d, $e \in \mathbb{Z}$, then* $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Proof. Note that d and e are odd, since they are relatively prime with $|\sigma_1|$. Hence the hypothesis (H4) and (H5) hold. Now $g^{-1}\sigma_1 g = \sigma_1^{e^{|g|-1}}$. Then $\rho(g^{-1}\sigma_1 g) = -\operatorname{id}$ and (H6) holds. The proof of (H7) is similar. $\square$

Example 4.13. *Let $m \geq 8$. Let $\sigma \in \mathbb{S}_m$ of type $(1^{n_1}, 2^{n_2}, 8^{n_8})$, with $n_8 \geq 1$, $\mathcal{O}$ the conjugacy class of σ and $\rho \in \widehat{\mathbb{S}_m^\sigma}$. Then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.*

Proof. By Lemma 1.3, we may suppose that $q_{\sigma\sigma} = -1$. If $n_8 \geq 3$, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$, from Corollary 3.10. We consider two cases.

CASE (I): $n_8 = 1$. Let $A_8 = (i_1\ i_2\ i_3\ i_4\ i_5\ i_6\ i_7\ i_8)$ the 8-cycle appearing in the decomposition of σ as product of disjoint cycles. We set $\alpha := \sigma\, A_8^{-1}$ and define $\sigma_1 := \sigma$, $\sigma_6 := \sigma_1^3$, $\tau_1 := \sigma_1^5$, $\tau_6 := \sigma_1^{-1}$,

$$\sigma_2 := (i_1\ i_3\ i_8\ i_6\ i_5\ i_7\ i_4\ i_2)\,\alpha, \qquad \sigma_3 := (i_1\ i_8\ i_2\ i_7\ i_5\ i_4\ i_6\ i_3)\,\alpha,$$
$$\sigma_4 := (i_1\ i_6\ i_4\ i_3\ i_5\ i_2\ i_8\ i_7)\,\alpha, \qquad \sigma_5 := (i_1\ i_7\ i_6\ i_8\ i_5\ i_3\ i_2\ i_4)\,\alpha,$$
$$\tau_2 := (i_1\ i_7\ i_8\ i_2\ i_5\ i_3\ i_4\ i_6)\,\alpha, \qquad \tau_3 := (i_1\ i_4\ i_2\ i_3\ i_5\ i_8\ i_6\ i_7)\,\alpha,$$
$$\tau_4 := (i_1\ i_2\ i_4\ i_7\ i_5\ i_6\ i_8\ i_3)\,\alpha, \qquad \tau_5 := (i_1\ i_3\ i_6\ i_4\ i_5\ i_7\ i_2\ i_8)\,\alpha.$$

CASE (II): $n_8 = 2$. Let

$$A_{1,8} = (i_1\ i_2\ i_3\ i_4\ i_5\ i_6\ i_7\ i_8) \ \text{ and } \ A_{2,8} = (i_9\ i_{10}\ i_{11}\ i_{12}\ i_{13}\ i_{14}\ i_{15}\ i_{16})$$

the two 8-cycles appearing in the decomposition of σ as product of disjoint cycles. We call $A_8 = A_{1,8}A_{2,8}$, $\alpha := \sigma\, A_8^{-1}$ and define $\sigma_1 := \sigma$, $\sigma_6 := \sigma_1^3$, $\tau_1 := \sigma_1^5$, $\tau_6 := \sigma_1^{-1}$,

$$\sigma_2 := (i_1\ i_3\ i_8\ i_6\ i_5\ i_7\ i_4\ i_2)(i_9\ i_{11}\ i_{16}\ i_{14}\ i_{13}\ i_{15}\ i_{12}\ i_{10})\,\alpha,$$
$$\sigma_3 := (i_1\ i_8\ i_2\ i_7\ i_5\ i_4\ i_6\ i_3)(i_9\ i_{16}\ i_{10}\ i_{15}\ i_{13}\ i_{12}\ i_{14}\ i_{11})\,\alpha,$$
$$\sigma_4 := (i_1\ i_6\ i_4\ i_3\ i_5\ i_2\ i_8\ i_7)(i_9\ i_{14}\ i_{12}\ i_{11}\ i_{13}\ i_{10}\ i_{16}\ i_{15})\,\alpha,$$
$$\sigma_5 := (i_1\ i_7\ i_6\ i_8\ i_5\ i_3\ i_2\ i_4)(i_9\ i_{15}\ i_{14}\ i_{16}\ i_{13}\ i_{11}\ i_{10}\ i_{12})\,\alpha,$$
$$\tau_2 := (i_1\ i_7\ i_8\ i_2\ i_5\ i_3\ i_4\ i_6)(i_9\ i_{15}\ i_{16}\ i_{10}\ i_{13}\ i_{11}\ i_{12}\ i_{14})\,\alpha,$$
$$\tau_3 := (i_1\ i_4\ i_2\ i_3\ i_5\ i_8\ i_6\ i_7)(i_9\ i_{12}\ i_{10}\ i_{11}\ i_{13}\ i_{16}\ i_{14}\ i_{15})\,\alpha,$$
$$\tau_4 := (i_1\ i_2\ i_4\ i_7\ i_5\ i_6\ i_8\ i_3)(i_9\ i_{10}\ i_{12}\ i_{15}\ i_{13}\ i_{14}\ i_{16}\ i_{11})\,\alpha,$$
$$\tau_5 := (i_1\ i_3\ i_6\ i_4\ i_5\ i_7\ i_2\ i_8)(i_9\ i_{11}\ i_{14}\ i_{12}\ i_{13}\ i_{15}\ i_{10}\ i_{16})\,\alpha.$$

In both cases, $\sigma_6 = \sigma_1^3$ and $\tau_1 = \sigma_1^5$ and $(\sigma, \tau) \subseteq \mathcal{O}$ is of type $\mathfrak{D}^{(2)}$. Then the result follows from Corollary 4.12. $\square$

Remarks 4.14. (i). The discussion in the preceding example can be adapted to $\sigma \in \mathbb{S}_m$ of type $(1^{n_1}, 2^{n_2}, \dots, m^{n_m})$ provided that $n_8 \geq 1$; but then some requirements on the representation ρ have to be imposed.

(ii). Let $N = 2^n$ with $n \geq 4$. It can be shown that the orbit of the N-cycle in $\mathbb{S}_N$ contains no family of type $\mathfrak{O}$ using Lemma 4.6.

(iii). The orbit with label $j = 4$ of the Mathieu group M_{22} contains a family of type $\mathfrak{O}^{(2)}$, and therefore this group admits no finite-dimensional pointed Hopf algebra except the group algebra itself [F1].

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